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# A Method for the 2-D Quasi-Isometric Regular Grid Generation

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A method for the generation of quasi-isometric boundary-fitted curvilinear coordinate systems for arbitrary domains is developed on the basis of the theory of conformal, quasi-conformal, and quasi-isometric mappings and results from the non-Euclidean geometry concerning surfaces of constant curvature. The method as it is proposed has an advantage over similar methods developed earlier in that the number of unknown parameters to be found is decreased, strict boundaries for parameters are found, and a simple and efficient process of identification of an unknown parameter is given. The reliability of the method is assured by an existence and uniqueness theorem for quasi-isometric maps between physical regions and geodesic quadrangles on surfaces of constant curvature which are used to constrict quasi-isometric grids in physical domains. We formulate the Riemannian metric consistent with this theorem which is available analytically. Illustrations of this technique are given for various domains. 

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## 1. INTRODUCTION

### 1.1. Quasi-Conformal Grids

The problem of generation of a structured grid in some physical domain $\mathcal{D}$ can be considered as a problem of construction of the mapping

$$X = X(\xi, \eta), \qquad Y = Y(\xi, \eta) \tag{1.1}$$

between the points $(\xi, \eta)$ of the regular computational region

$$\mathcal{R} = \{(\xi, \eta) : 0 \leq \xi \leq 1, 0 \leq \eta \leq 1\}$$

and the points $(X, Y)$ of the given physical domain $\mathcal{D}$ with interiors angles $\beta_i$, $0 < \beta_i < \pi$, $i = 1, \ldots, 4$.



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Considerable progress has been made in the development of numerically generated coordinate systems, and a variety of generating systems have been presented in the literature [8, 12, 22, 24].

Among different approaches to the problem stated above we can emphasize the two most commonly known which use the elliptic generating systems. The first approach was proposed in 1966 by Winslow [25] and the core of the method consisted of application of the theory of two-dimensional harmonic mappings to the problem. As was shown in 1978 by Mastin and Thompson [19], the mapping generated by the Winslow system has a non-vanishing Jacobian.

Another approach using conformal mappings of a rectangular region

$$\mathcal{Q} = \left\{(\xi, \eta) : 0 \leq \xi \leq 1/\sqrt{\mathcal{M}}, 0 \leq \eta \leq \sqrt{\mathcal{M}}\right\}$$

with an *a priori* given grid onto the physical region was proposed in 1967 by Godunov and Prokopov [13]. Here $\mathcal{M}$ is the conformal modulus of $\mathcal{D}$, which guarantees the existence and the uniqueness of the conformal mapping sought. The development of this method is precisely described in the book by Godunov *et al.* [12] and the survey by Thompson *et al.* [22]. The orthogonal mapping technique has been investigated in recent works [16] and [7, 20] from different points of view.

In 1975 Belinsky *et al.* [1] proposed to use a quasi-conformal mapping of the unit square onto the given physical region instead of a conformal one. The mapping was to be found a composition of two mappings $\mathcal{R} \to \mathcal{P} \to \mathcal{D}$, namely, a Chebyshev mapping

$$x = x(\xi, \eta), \qquad y = y(\xi, \eta) \tag{1.2}$$

of the unit square $\mathcal{R}$ onto a plane curvilinear parallelogram $\mathcal{P}$, which generates the Riemmanian metric

$$ds^2 = g_{11} d\xi^2 + 2g_{12}\, d\xi\, d\eta + g_{22}\, d\eta^2, \tag{1.3}$$

where $g_{11} = x_\xi^2 + y_\xi^2$, $g_{12} = x_\xi x_\eta + y_\xi y_\eta$, $g_{22} = x_\eta^2 + y_\eta^2$, and a conformal mapping of $\mathcal{P}$ onto the physical region $\mathcal{D}$. The coordinate systems (1.1) were proposed to be sought as solutions of variational problems. On the other hand a superposition of these mappings can be considered as a solution of the Beltrami system

$$gX_\xi = -g_{12} Y_\xi + g_{11} Y_\eta, \qquad gX_\eta = -g_{22} Y_\xi + g_{12} Y_\eta, \qquad g^2 = g_{11} g_{22} - g_{12}^2. \tag{1.4}$$

The system (1.4) is a generalization of the Cauchy–Riemann equations and can be treated [3, 17] as a condition for conformality of the mapping (1.1) with respect to the Riemmanian metric (1.3); that is, the mapping (1.1) that satisfies (1.4) maps every two curves in the $(\xi, \eta)$-plane which make the angle $\alpha$ at the point of their intersection *measured in the metric* (1.3) into curves in the $(X, Y)$ domain which make the same angle $\alpha$. This point of view is interesting in the sense that by defining the type of $g_{ij}$ explicitly we can control the quality of the grid by controlling the angle $g_{12}(g_{11}g_{22})^{-1/2} = \cos\theta$ between the lines, and the ratio of cell sides $(g_{22}/g_{11})^{1/2}$ as well.

In such a way in the paper [11] it was proposed to use the certain class of functions $g_{ij}$ depending on $\xi$, $\eta$ and unknown vector of parameters $r$ for the process of construction of structured multi-block quasi-conformal grids in complex domains.

However, in the case of quasi-conformal mappings, each of which is a composition of a conformal and some other mapping, for example, an algebraic transformation [21], we are not practically able to control the size of cells as the grid is refined. This fact is connected with the behavior of the modulus of a conformal mapping's derivative on the boundaries of the domain. As a rule, in the corners of the mapped domain the modulus of the conformal mapping's derivative approaches either zero of infinity [14, 18]. In other words, the Jacobian of the transformation has a singular value that approaches either zero or infinity as the grid is refined.

In order to have better control of the grid cells quality, we have to restrict the class of coefficients $g_{ij}$ of the metric (1.3) in such a way that the solution of the corresponding Beltrami system is $\mu$-quasi-isometric. Under a $\mu$-quasi-isometric mapping (1.1) an infinitesimal square will go over into a parallelogram with sides of the length $\Delta S_\xi$ and $\Delta S_\eta$, which are connected to the changes in $\xi$ and $\eta$ in the following way,

$$\Delta S_\xi = G_{11}^{1/2}\Delta\xi, \qquad \Delta S_\eta = G_{22}^{1/2}\Delta\eta,$$

where $G_{11} = X_\xi^2 + Y_\xi^2$, $G_{12} = X_\xi X_\eta + Y_\xi Y_\eta$, $G_{22} = X_\eta^2 + Y_\eta^2$; moreover the following estimates hold:

$$\frac{\Delta\xi}{\mu} \le \Delta S_\xi \le \mu\Delta\xi, \qquad \frac{\Delta\eta}{\mu} \le \Delta S_\eta \le \mu\Delta\eta.$$

In other words, singular values of the Jacobian of a $\mu$-quasi-isometric mapping are bounded from above and below by $\mu$ and $1/\mu$, which gives us no singularities in corners of the physical domain as the grid is refined.

### 1.2. Quasi-Isometric Grids

The generation of 2-D quasi-isometric grids may be considered as the following boundary value problem (BVP): given a quasi-isometric mapping between $\partial\mathcal{R}$ and $\partial\mathcal{D}$ to extend it inside $\mathcal{R}$ as a quasi-isometric solution of the Beltrami system (1.4) with appropriate $g_{ik}$ from a given class of coefficients. Boundary conditions in this BVP are either Dirichlet conditions with fixed boundary points or "free" conditions under which grid points on the boundary of the physical region $\mathcal{D}$ are not fixed and can move along $\partial\mathcal{D}$.

In order to obtain a quasi-isometric solution of the grid generation problem, a special one-parametric family of metrics (1.3) was studied by one of the authors in [5, 6]. In a later paper by Godunov *et al.* [10] it was proposed to study a special five-parameter family of metrics. However, an identification process of the unknown parameters was extremely difficult because the domain of the five parameters was defined implicitly.

The present paper is aimed to develop ideas introduced in [5, 6]. The main goals are to describe a one-parametric family of coefficients $g_{ik}$ with a parameter $r$, for which the posed BVP has the unique quasi-isometric solution; to determine precise bounds $r^{\min}$ and $r^{\max}$ for $r$; and to develop a new technique for finding of the unknown parameter.

We construct the mapping (1.1) as a composition of two quasi-isometric mappings [10].

The first transformation maps the computational region $\mathcal{R}$ onto some geodesic quadrangle $\mathcal{P}$ with the angles $\alpha_1, \ldots, \alpha_4$ on a surface of constant curvature. The geodesic quadrangle $\mathcal{P}$ with angles $\alpha_j$ is to be chosen in such a way that $\alpha_j$ coincide with corresponding angles $\beta_j$ of the physical domain $\mathcal{D}$, and conformal modules of $\mathcal{P}$ and $\mathcal{D}$ are the same. Under the condition

$$\sum_{j=1}^{4} \beta_j - 2\pi < 2\beta_i, \qquad i = 1, \dots, 4, \tag{1.5}$$

such a geodesic quadrangle $\mathcal{P}$ exists uniquely in both cases of the negative [5, 10] and positive [6] angle defect of $\mathcal{D}$. After constructing the quadrangle $\mathcal{P}$ we generate a geodesic grid in $\mathcal{P}$ by means of geodesic bundles as suggested in [5]. In other words, the geodesic grids in $\mathcal{P}$ can be treated as a variant of the Winslow grids with an advantage that in our case the grid in $\mathcal{P}$ can be defined explicitly and still will posses all the attractive features of the Winslow grids. In contrast to the work [10], which uses conformal and projective mappings, we use a conformal representation of spherical and hyperbolic geometries in order to construct a geodesic grid in $\mathcal{P}$. In this way we have direct information about domain angles which becomes implicit if we use projective mappings; this allows us to reduce the number of parameters to be determined to one instead of five, as it was in [10].

By the second mapping $\mathcal{P}$ is mapped conformally onto the physical domain $\mathcal{D}$; such a conformal mapping exists uniquely by virtue of the Riemann Mapping Theorem [15], and if we assume that all sides of $\mathcal{D}$ are smooth enough then the mapping of $\mathcal{P}$ onto $\mathcal{D}$ will be quasi-isometric [14, 18] as well.

The composite mapping is to be found as follows: The first quasi-isometric mapping generates the metric tensor $g_{ij}$, and the elements of the metric tensor are used as coefficients of the Beltrami system (1.4). By solving the system we obtain the quasi-isometric mapping (1.1) sought. The problem of determining the metric tensor $g_{ij}$ and functions (1.1) for which (1.4) holds can be formulated as a variational problem of minimizing the functional of Dirichlet type.

This method can be used for the generation of quasi-conformal grids in the physical domain $\mathcal{D}$ with the angles $\beta_1, \dots, \beta_4$ which do not satisfy the inequality (1.5), for example, when the boundary $\partial\mathcal{D}$ is a smooth closed curve. For this purpose it is sufficient to define angles $\alpha_1, \dots, \alpha_4$ of $\mathcal{P}$ which satisfy the condition (1.5). Angles $\alpha_j$ define the internal grid angle and might not coincide with the real angles of the domain.

Thus the new contribution of our work is the class of functions $g_{11}, g_{22}, g_{12}$ of the independent variables $\xi, \eta, \alpha = (\alpha_1, \dots, \alpha_4)$ and of unknown parameter $r$ (a monotonic function of $\mathcal{M}$ which ranges from $r^{\min}$ to $r^{\max}$), for the generation of quasi-isometric grids and the new technique for finding the unknown parameter $r$ of the mapping sought.

The main advantage of the proposed method is that under certain conditions on $g_{ij}$ the mentioned quasi-isometric mapping of $\mathcal{R}$ onto $\mathcal{D}$ is proved to exist uniquely and the conformally equivalent metrics induced by the mapping (1.1) are available analytically, which reduces the tome of actual computing. Moreover, our method in the form as it is proposed in the present paper provides a certain flexibility in the sense that two cases of the boundary points behavior are admitted: they might be chosen fixed or may move along the boundary; the method allows more direct control of the grid cells size and quality as the grids are refined, which is important for finite-difference numerical methods used in computational physics, e.g., multigrid [9].

## 2. GEOMETRY OF SURFACES OF CONSTANT CURVATURE

### 2.1. Geodesic Lines and Geodesic Bundles

In order to provide a conformal representation of spherical, Euclidean, and hyperbolic geometries we will consider the surface of constant curvature $K = 4\delta$ as the plane $(x, y)$

with the metric

$$ds^2 = \frac{dx^2 + dy^2}{(1 + \delta(x^2 + y^2))^2}, \tag{2.1}$$

where $\delta$ is a real number [4]. If $\delta$ is positive, the metric (2.1) is defined for every $x$ and $y$ including infinite, and we obtain the representation of spherical geometry. If $\delta$ is negative, then in the circle $x^2 + y^2 + \delta < 0$ we obtain one of the possible Lobachevsky geometry representations, or hyperbolic geometry. In the case $\delta = 0$, the metric (2.1) is ordinary Euclidean metric on the plane $(x, y)$.

Geodesics in the metric (2.1) are curves defined by the equation

$$ax + by + c[1 - \delta(x^2 + y^2)] = 0. \tag{2.2}$$

When $a^2 + b^2 + 4c^2\delta > 0$, Eq. (2.2) defines a straight line or a circle. If $a^2 + b^2 + 4c^2\delta < 0$, then the set of points satisfying (2.2) can be considered to be circle of imaginary non-zero radius. In the case $c\delta = 0$ we have straight lines on the plane $(x, y)$, and if $a = b = 0$ and $c \neq 0$ then the set defined by (2.2) is the line at infinity.

Let $q = ax + by + c[1 - \delta(x^2 + y^2)]$. Consider the fundamental quadratic form of coefficients $a, b, c, c\delta$,

$$\rho(q) = R(q, q) = \frac{a^2 + b^2 + 4c^2\delta}{4}, \tag{2.3}$$

and its polarization

$$R(q_1, q_2) = \frac{1}{4}[a_1a_2 + b_1b_2 + 2c_1c_2(\delta_1 + \delta_2)]. \tag{2.4}$$

Let $s_1$ and $s_2$ be two distinct geodesics defined by equations $q_1 = 0$ and $q_2 = 0$, respectively. The condition

$$\bar{R}(q_1, q_2) \equiv \frac{R(q_1, q_2)}{\sqrt{\rho(q_1)\rho(q_2)}} \leq 1 \tag{2.5}$$

is necessary and sufficient for existence of real points of intersection of circles $s_1$ and $s_2$ [2]. The angle between $s_1$ and $s_2$ is a real number $\alpha(s_1, s_2)$ determined by

$$\alpha(s_1, s_2) = \begin{cases} \arccos \bar{R}(q_1, q_2), & \text{if } \bar{R}(q_1, q_2) \leq 1, \\ \operatorname{arccosh} \bar{R}(q_1, q_2), & \text{if } \bar{R}(q_1, q_2) > 1, \end{cases} \tag{2.6}$$

The formula (2.6) implies that for every $\delta$ the circles of the form (2.2) are orthogonal to the circle $1 + \delta(x^2 + y^2) = 0$, which is called the absolute.

The family $\mathcal{F}$ of geodesics orthogonal to $s_1$ and $s_2$ is called a geodesic bundle. The geodesic bundle $\mathcal{F}^{\perp}$ in which $s_1$ and $s_2$ can be embedded is called orthogonal to $\mathcal{F}$.

Note that if $\delta \neq 0$ and $\bar{R}(q_1, q_2) \leq 1$, then geodesics from $\mathcal{F}$ do not have common points and $\mathcal{F}$ is called a hyperbolic bundle of geodesics, and if $\bar{R}(q_1, q_2) > 1$, then geodesics from $\mathcal{F}$ have exactly two common points and $\mathcal{F}$ is called elliptic.

### 2.2. The Group of Motions

We consider three types of non-Euclidean spaces of a constant curvature indicated above. All of them admit the continuous group of isometric mappings, that is, the group of motions. If we consider the parametric plane as a complex plane, a motion can be represented as a linear-fractional transformation of a special form. If we denote $x + iy$ by $z$, then every motion has the form

$$w(z) = e^{i\omega} \frac{z - \zeta}{1 + \delta \bar{\zeta} z}, \tag{2.7}$$

where $\omega \in R$ and the complex number $\zeta$ must satisfy the following condition: If $\delta < 0$ then $|\zeta| < |\delta|^{-1/2}$, otherwise $\zeta$ can be any complex number. In other words, each linear-fractional mapping of the form (2.7) maps the absolute onto itself and does not change the differential increment of non-Euclidean arc length, i.e.,

$$\frac{|dw(z)|}{1 + \delta |w(z)|^2} = \frac{|dz|}{1 + \delta |z|^2}.$$

### 2.3. Geodesic Quadrangles and Characteristic Invariants

Let $\mathcal{P}$ be a quadrangle, sides of which lie on geodesics (2.2), and let its vertices $(z_i)_{i=1,\ldots,4}$ be enumerated counterclockwise. Let us denote sides of $\mathcal{P}$ by $A_i = z_i z_{i+1}$, and angles between $A_{i-1}$ and $A_i$ by $\alpha_i$, $i = 1, \ldots, 4$ ($z_5 \equiv z_1$ and $A_0 \equiv A_4$). Let $\varphi_i = \alpha_i - \pi/2$, $i = 1, \ldots, 4$.

It is possible by means of linear-fractional transformation of the form (2.7) to put a geodesic quadrangle $\mathcal{P}$ into the "standard" position, i.e., to move one vertex of $\mathcal{P}$ (say, $z_i$) to the origin and rotate $\mathcal{P}$ so that the side $A_i$ will be a segment of the positive $x$-axis. Denote by $r_i$ the Euclidean length of the segment $A_i$. The invariance of the metric (2.1) under the motions (2.7) implies that we can associate with every $A_i$ its *Euclidean length* $r_i$ which is uniquely defined.

The question may arise about a characteristic of $\mathcal{P}$ that is necessary and sufficient for distinguishing two geodesic quadrangles with the same angles. In the capacity of such a characteristic we propose the invariant

$$m(\mathcal{P}) = \frac{r_1 r_3}{r_2 r_4}. \tag{2.8}$$

From the results presented in [5, 6] it follows that the quantity $m(\mathcal{P})$ depends monotonically on the conformal module $\mathcal{M}(\mathcal{P})$. In other words, the following theorem holds:

THEOREM 1. *Let $\mathcal{P}$ and $\tilde{\mathcal{P}}$ be two geodesic quadrangles such that $\alpha_i = \tilde{\alpha}_i$, $i = 1, \ldots, 4$. If $r_1 = \tilde{r}_1$ then $\mathcal{P} = \tilde{\mathcal{P}}$, if $r_1 > \tilde{r}_1$ then $m(\mathcal{P}) > m(\tilde{\mathcal{P}})$ and $\mathcal{M}(\mathcal{P}) > \mathcal{M}(\tilde{\mathcal{P}})$.*

## 3. GEODESIC GRIDS IN CONVEX GEODESIC QUADRANGLES ON THE PLANE

In this section we shall develop the technique of embedding an arbitrary geodesic quadrangle $\mathcal{P}$ on the plane into one-parametric family of quadrangles $\mathcal{P}_{r_1}$ which have the same angles, and construct a mapping which gives us geodesic grid in any quadrangle from the family.

### 3.1. Constriction of a Geodesic Quadrangle with Given Properties

Note that in this “flat” case the previously defined quantity $\delta$ is zero, and the angle defect of $\mathcal{P}$ is also zero, which means $\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 = 0$. We can treat the $(x, y)$-plane as a surface with constant curvature zero and apply all results mentioned in the previous section. Since $\delta = 0$, a geodesic in the metric (2.1) is a straight line defined by the equation $ax + by + c = 0$.

Since a geodesic quadrangle is uniquely determined by a set of five parameters, in order to obtain a one-parametric family of quadrangles we have to fix four parameters, for example, three angles and the area of quadrangle. We shall choose for the varying parameter the Euclidean length $r_1$ of the side $A_1$. As we will show later, the parameter $r_1$ can vary between the boundaries $r_1^{\min}$ and $r_1^{\max}$, and $\mathcal{M}(\mathcal{P}_{r_1}) \to 0$ as $r_1 \to r_1^{\min}$, and $\mathcal{M}(\mathcal{P}_{r_1}) \to \infty$ as $r_1 \to r_1^{\max}$.

Let the left side $A_4$ and the right side $A_2$ of $\mathcal{P}_{r_1}$ lie on the lines

$$\cos\varphi_1 x + \sin\varphi_1 y = 0 \qquad \text{and} \qquad \cos\varphi_2 x - \sin\varphi_2 y - r_1 \cos\varphi_2 = 0.$$

Let the lower side $A_1$ and the upper side $A_3$ of $\mathcal{P}_{r_1}$ lie on the lines which have the equations

$$-y = 0 \qquad \text{and} \qquad \sin(\varphi_1 + \varphi_4)\, x - \cos(\varphi_1 + \varphi_4) y + r_4 \cos\varphi_4 = 0.$$

So far we have considered $r_4$ as an independent parameter; later we will define $r_4 = r_4(r_1)$ as a function of $r_1$.

### 3.2. Geodesic Grids in Geodesic Quadrangles

Let us consider a pencil of lines $\mathcal{F}_\xi$, depending on a parameter $\xi \in [0, 1]$, such that $A_4$ lies on the line $\xi = 0$ from $\mathcal{F}_\xi$, and $A_2$ lies on the line $\xi = 1$ from $\mathcal{F}_\xi$. Elements of the pencil $\mathcal{F}_\xi$ are described by the equation

$$[\cos\varphi_1 + \xi(\cos\varphi_2 - \cos\varphi_1)]\, x + [\sin\varphi_1 - \xi(\sin\varphi_2 + \sin\varphi_1)]\, y - \xi r_1 \cos\varphi_2 = 0, \qquad \xi \in [0, 1]. \tag{3.1}$$

Now consider a pencil of lines $\mathcal{F}_\eta$, depending on a parameter $\eta \in [0, 1]$, such that $A_1$ lies on the line $\eta = 0$ from $\mathcal{F}_\eta$, and $A_3$ lies on the line $\eta = 1$ from $\mathcal{F}_\eta$. So, all elements of $\mathcal{F}_\eta$ satisfy the equation

$$\eta \sin(\varphi_1 + \varphi_4)\, x - [1 + \eta(\cos(\varphi_1 + \varphi_4) - 1)] y + \eta r_4 \cos\varphi_4 = 0, \qquad \eta \in [0, 1]. \tag{3.2}$$

From Eqs. (3.1)–(3.2) we can obtain the following mapping of the unit square $\mathcal{R}$ on the $(\xi, \eta)$-plane onto the quadrangle $\mathcal{P}_{r_1}$,

$$x = [\xi c_1 - \eta c_2 \sin\varphi_1 + \xi\eta(b_1 c_2 + b_2 c_1)]/\mathcal{C}(\xi, \eta), \tag{3.3}$$

$$y = \eta[c_2 \cos\varphi_1 + \xi(a_1 c_2 + a_2 c_1)]/\mathcal{C}(\xi, \eta), \tag{3.4}$$

where

$$a_1 = \cos\varphi_2 - \cos\varphi_1, \qquad b_1 = \sin\varphi_1 + \sin\varphi_2, \qquad c_1 = r_1 \cos\varphi_2,$$

$$a_2 = \sin(\varphi_1 + \varphi_4), \qquad b_2 = \cos(\varphi_1 + \varphi_4) - 1, \qquad c_2 = r_4 \cos\varphi_4,$$

$$\mathcal{C}(\xi, \eta) = \cos\varphi_1 - \xi(\cos\varphi_1 - \cos\varphi_2) - \eta(\cos\varphi_1 - \cos\varphi_4) + \xi\eta(\cos\varphi_1 - \cos\varphi_2 + \cos\varphi_3 - \cos\varphi_4).$$

Note that in (3.1) and (3.2) instead of $\xi$ and $\eta$ we can take two arbitrary monotone increasing functions $\xi(\xi_1)$ and $\eta(\eta_1)$ satisfying the conditions $\xi(0) = \eta(0) = 0$ and $\xi(1) = \eta(1) = 1$. In particular, we can choose $\xi = \xi(\xi_1)$ and $\eta = \eta(\eta_1)$ in such a way that under the mapping (3.3)–(3.4) the uniform distribution of points on the lower and the left sides of the unit square holds; i.e., the distribution of points on sides $A_1$ and $A_4$ of the quadrangle $\mathcal{P}_{r_1}$ is also uniform. In order to obtain such a mapping it is sufficient to choose $\xi(\xi_1)$ and $\eta(\eta_1)$ as

$$\xi(\xi_1) = \frac{\xi_1 \cos\varphi_1}{\xi_1 \cos\varphi_1 + (1-\xi_1)\cos\varphi_2}, \qquad \eta(\eta_1) = \frac{\eta_1 \cos\varphi_1}{\eta_1 \cos\varphi_1 + (1-\eta_1)\cos\varphi_4}. \quad (3.5)$$

### 3.3. One-Parametric Family of Geodesic Quadrangles

No we are going to get rid of the dependent parameter $r_4$. So far we have fixed only four parameters—angles $\alpha_j$ of the geodesic quadrangle $\mathcal{P}_{r_1}$. We choose the area of $\mathcal{P}_{r_1}$ to be the fifth parameter to fix. If we assume that the area of quadrangle $\mathcal{P}_{r_1}$ is equal to 1/2, then parameters $r_1$ and $r_4$ satisfy an algebraic equation of the form

$$\cos\varphi_3 = \cos\varphi_2 \sin(\varphi_1+\varphi_4) r_1^2 + \cos\varphi_4 \sin(\varphi_1+\varphi_2) r_4^2 + 2\cos\varphi_2 \cos\varphi_4 r_1 r_4. \quad (3.6)$$

Note that Eq. (3.6) is given in implicit form, but regarding $r_4$ as the dependent variable, we can obtain the explicit form of the function $r_4 = r_4(r_1)$ with the domain $r_1^{\min} < r_1 < r_1^{\max}$. For this purpose we first rewrite (3.6) as

$$\frac{1}{r_4^2} - 2B_0\frac{1}{r_4} + C_0 = 0, \quad (3.7)$$

where

$$B_0 = \frac{\cos\varphi_2 \cos\varphi_4 r_1}{\cos\varphi_3 - \cos\varphi_2 \sin(\varphi_1+\varphi_4) r_1^2}, \qquad C_0 = \frac{-\cos\varphi_4 \sin(\varphi_1+\varphi_2)}{\cos\varphi_3 - \cos\varphi_2 \sin(\varphi_1+\varphi_4) r_1^2}.$$

Since we need to find the positive root $r_4 = r_4(r_1)$ when $C_0 < 0$, and the least positive root when $C_0 \geq 0$, we have

$$r_4(r_1) = \frac{1}{B_0 + \sqrt{B_0^2 - C_0}}. \quad (3.8)$$

Moreover, we can calculate the derivative

$$\frac{dr_4(r_1)}{dr_1} = -\frac{r_1 \cos\varphi_2 \sin(\varphi_1+\varphi_4) + r_4(r_1)\cos\varphi_2\cos\varphi_4}{r_1\cos\varphi_2\cos\varphi_4 + r_4(r_1)\cos\varphi_4 \sin(\varphi_1+\varphi_2)}. \quad (3.9)$$

### 3.4. Strict Boundaries for $r_j$, Plane Case

Let us denote by $\mathcal{S}_4$ the symmetric group of permutations of the set {1, 2, 3, 4} [23]. Let $\Sigma_4$ be the cyclic subgroup of $\mathcal{S}_4$ generated by the element

$$\bar{\sigma} \in \mathcal{S}_4, \qquad \bar{\sigma} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}.$$

Let $\sigma \in \Sigma_4$ be such that

$$\varphi_{\sigma(1)} + \varphi_{\sigma(2)} \geq 0, \qquad \varphi_{\sigma(1)} + \varphi_{\sigma(4)} \geq 0.$$

Then from (3.6) we can derive the strict boundaries for the values of Euclidean lengths of sides of $\mathcal{P}_{r_1}$ (provided the angles are fixed),

$$r_{\sigma(i)}^{\min} < r_{\sigma(i)} < r_{\sigma(i)}^{\max}, \qquad i = 1, \dots, 4,$$

where

$$\begin{aligned}
r_{\sigma(1)}^{\min} &= 0, & r_{\sigma(1)}^{\max} &= \sqrt{\frac{\cos\varphi_{\sigma(3)}}{\cos\varphi_{\sigma(2)}\sin\left(\varphi_{\sigma(1)}+\varphi_{\sigma(4)}\right)}},\\
r_{\sigma(2)}^{\min} &= \sqrt{\frac{\sin\left(\varphi_{\sigma(1)}+\varphi_{\sigma(4)}\right)}{\cos\varphi_{\sigma(2)}\cos\varphi_{\sigma(3)}}}, & r_{\sigma(2)}^{\max} &= \sqrt{\frac{\cos\varphi_{\sigma(4)}}{\cos\varphi_{\sigma(3)}\sin\left(\varphi_{\sigma(1)}+\varphi_{\sigma(2)}\right)}},\\
r_{\sigma(3)}^{\min} &= \sqrt{\frac{\sin\left(\varphi_{\sigma(1)}+\varphi_{\sigma(2)}\right)}{\cos\varphi_{\sigma(3)}\cos\varphi_{\sigma(4)}}}, & r_{\sigma(3)}^{\max} &= \sqrt{\frac{\cos\varphi_{\sigma(2)}}{\cos\varphi_{\sigma(3)}\sin\left(\varphi_{\sigma(1)}+\varphi_{\sigma(4)}\right)}},\\
r_{\sigma(4)}^{\min} &= 0, & r_{\sigma(4)}^{\max} &= \sqrt{\frac{\cos\varphi_{\sigma(3)}}{\cos\varphi_{\sigma(4)}\sin\left(\varphi_{\sigma(1)}+\varphi_{\sigma(2)}\right)}}.
\end{aligned} \tag{3.10}$$

### 3.5. Riemannian Metric Induced by the Quasi-Isometric Parameterization

Now formulas (3.3)–(3.4) and (3.8) provide us with a quasi-isometric parameterization of a geodesic quadrangle $\mathcal{P}_{r_1}$ for any $r_1 \in [r_1^{\min}, r_1^{\max}]$. The parameterization (3.3)–(3.4) induces a class of conformally equivalent Riemmanian metrics on $\mathcal{P}_{r_1}$ of the form

$$ds^2 = g_{11}(\xi,\eta,r_1)\,d\xi^2 + 2g_{12}(\xi,\eta,r_1)\,d\xi\,d\eta + g_{22}(\xi,\eta,r_1)\,d\eta^2. \tag{3.11}$$

In this section we will find elements of the metric tensor $g_{11} = x_\xi^2 + y_\xi^2$, $g_{12} = x_\xi x_\eta + y_\xi y_\eta$, and $g_{22} = x_\eta^2 + y_\eta^2$ in the explicit form.

We can calculate all first derivatives of the mapping (3.3)–(3.4) explicitly,

$$\begin{aligned}
x_\xi &= (1+\eta b_2)\cdot\mathcal{A}(\eta,r_1)/\mathcal{C}^2(\xi,\eta),\\
y_\xi &= \eta a_2\cdot\mathcal{A}(\eta,r_1)/\mathcal{C}^2(\xi,\eta),\\
x_\eta &= (b_1\xi - \sin\varphi_1)\cdot\mathcal{B}(\xi,r_1)/\mathcal{C}^2(\xi,\eta),\\
y_\eta &= (\cos\varphi_1 + \xi a_1)\cdot\mathcal{B}(\xi,r_1)/\mathcal{C}^2(\xi,\eta),
\end{aligned} \tag{3.12}$$

where

$$\mathcal{A}(\eta,r_1) = r_1\cos\varphi_2[\eta\cos\varphi_4 + (1-\eta)\cos\varphi_1] + \eta r_4(r_1)\cos\varphi_4\sin(\varphi_1+\varphi_2),$$

$$\mathcal{B}(\xi,r_1) = r_4(r_1)\cos\varphi_4[\xi\cos\varphi_2 + (1-\xi)\cos\varphi_1] + \xi r_1\cos\varphi_2\sin(\varphi_1+\varphi_4).$$

It is easily seen that for all $(\xi,\eta)$ in $\mathcal{R}$ the inequalities $\mathcal{A}(\eta,r_1) > 0$, $\mathcal{B}(\xi,r_1) > 0$, and $\mathcal{C}(\xi,\eta) > 0$ hold for all $r_1 \in (r_1^{\min}, r_1^{\max})$. The Jacobian of the transformation (3.3)–(3.4) is given by

$$\sqrt{g_{11}g_{22} - g_{12}} = \mathcal{A}(\eta,r_1)\mathcal{B}(\xi,r_1)/\mathcal{C}^3(\xi,\eta).$$

As a representative of the class of metrics (3.11) we can take the metric with the coefficients

$$\begin{aligned} g_{11}(\xi,\eta,r_1) &= \mathcal{A}^2(\eta,r_1)\{1+2\eta(1-\eta)[\cos(\varphi_1+\varphi_4)-1]\},\\ g_{22}(\xi,\eta,r_1) &= \mathcal{B}^2(\xi,r_1)\{1+2\xi(1-\xi)[\cos(\varphi_1+\varphi_2)-1]\},\\ g_{12}(\xi,\eta,r_1) &= \mathcal{A}(\eta,r_1)\mathcal{B}(\xi,r_1)\mathcal{S}(\xi,\eta), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} \mathcal{S}(\xi,\eta) = {}&-\sin\varphi_1+\xi(\sin\varphi_1+\sin\varphi_2)+\eta(\sin\varphi_1+\sin\varphi_4)\\ &-\xi\eta(\sin\varphi_1+\sin\varphi_2+\sin\varphi_3+\sin\varphi_4). \end{aligned}$$

The discriminant of the metric form is

$$g_{11}(\xi,\eta,r_1)g_{22}(\xi,\eta,r_1)-g_{12}^2(\xi,\eta,r_1)=\mathcal{A}^2(\eta,r_1)\mathcal{B}^2(\xi,r_1)\mathcal{C}^2(\xi,\eta).$$

## 4. GEODESIC GRIDS IN CONVEX GEODESIC QUADRANGLES ON A SURFACE OF CONSTANT CURVATURE

Our main goal is to construct a family $\mathcal{P}_{r_1}$ of geodesic quadrangles on a surface of constant curvature, each of which has same angles $\alpha_1,\dots,\alpha_4$, parameterized by $r_1$ in such a way that $\mathcal{M}(\mathcal{P}_{r_1})\to 0$ as $r_1\to r_1^{\min}$ and $\mathcal{M}(\mathcal{P}_{r_1})\to\infty$ as $r_1\to r_1^{\max}$. In general, we will repeat our argument for the "flat" case with slight modifications. First we shall obtain equations of geodesics that sides of the quadrangle belong to, taking as given five independent data—$\alpha_1,\alpha_2,\alpha_4,r_1$, and $r_4$—and later we will express $r_4$ as a function of $r_1,\alpha_1,\alpha_2,\alpha_3,\alpha_4$ and define $r_1^{\min}$ and $r_1^{\max}$.

Later in this section we will assume that the following conditions are satisfied:

$$r_1^{\min}<r_1<r_1^{\max},\qquad -\frac{\pi}{2}<\varphi_i<\frac{\pi}{2},\qquad \gamma<\frac{\pi}{2}+\varphi_i\qquad i=1,\dots,4. \tag{4.1}$$

### 4.1. Construction of Geodesic Quadrangles on Surfaces of Constant Curvature

Let us construct a geodesic quadrangle $\mathcal{P}=(z_i)_{i=1,\dots,4}$ with given parameters $\varphi_1,\varphi_2,\varphi_4$, $r_1$, and $r_4$. Assume that $z_1=(0,0)$ and the side $A_1$ is a segment of the positive $x$-axis and has Euclidean length $r_1$. Then the side $A_4$ on the parametric plane is represented by another segment that belongs to a ray going out of the origin at the angle $\alpha_1$ to the positive $x$-axis, and the vertex $z_4$ of the geodesic quadrangle $\mathcal{P}$ has coordinates $z_4=(-r_4\sin\varphi_1, r_4\cos\varphi_1)$. Assume that the side $A_3$ belongs to the geodesic of the form (2.2) that passes through the point $z_4$ and has tangent line at $z_4$ that intersects the $x$-axis at the angle $\varphi_1+\varphi_4$. Then by setting $c=r_4\cos\varphi_1$ and having $\delta$ fixed we obtain the following equation of the geodesic:

$$\begin{aligned} &\left[\sin(\varphi_1+\varphi_4)-\delta r_4^2\sin(\varphi_1-\varphi_4)\right]x-\left[\cos(\varphi_1+\varphi_4)-\delta r_4^2\cos(\varphi_1-\varphi_4)\right]y\\ &\quad+r_4\cos\varphi_4[1-\delta(x^2+y^2)]=0. \end{aligned} \tag{4.2}$$

Finally, assume that the side $A_2$ corresponds to an arc of the circle of the form (2.2) that passes through the point $z_2=(r_1,0)$. Let the tangent line to the geodesic at $z_2$ intersect the

$x$-axis at the angle $\pi/2 - \varphi_2$. By setting $c = r_1 \cos \varphi_2$ we obtain that the side $A_2$ belongs to the geodesic that satisfies the equation

$$\cos \varphi_2 \left(1 - \delta r_1^2\right) x - \sin \varphi_2 \left(1 + \delta r_1^2\right) y - r_1 \cos \varphi_2 [1 - \delta(x^2 + y^2)] = 0. \tag{4.3}$$

The closed figure formed by geodesic segments $A_1, \ldots, A_4$ is called the geodesic quadrangle $\mathcal{P}$. We shall consider only a convex geodesic quadrangle: This means that with any two points it also contains the geodesic segment between them.

### 4.2. Quasi-Isometric Parameterization of a Geodesic Quadrangle

In this section we construct a geodesic grid for $\mathcal{P}_{r_1}$ by an analytical quasi-isometric transformation of $\mathcal{R}$ onto $\mathcal{P}$.

Let the left and the right sides of a geodesic quadrangle $\mathcal{P}_{r_1}$ belong to the geodesics that satisfy $\cos \varphi_1 x + \sin \varphi_1 y = 0$ and Eq. (4.3), respectively. Then we consider every vertical grid line as an element of the geodesic bundle

$$\left(\cos \varphi_1 + \xi a_1^\delta\right) x + \left(\sin \varphi_1 - \xi b_1^\delta\right) y - \xi c_1 [1 - \delta(x^2 + y^2)] = 0, \qquad \xi \in [0, 1], \tag{4.4}$$

with

$$a_1^\delta = \left(1 - \delta r_1^2\right) \cos \varphi_2 - \cos \varphi_1, \qquad b_1^\delta = \left(1 + \delta r_1^2\right) \sin \varphi_2 + \sin \varphi_1, \qquad c_1 = r_1 \cos \varphi_2.$$

Further let the lower and the upper sides of the quadrangle belong to geodesics that satisfy $-y = 0$ and Eq. (4.2), respectively. Then we consider every horizontal grid line as an element of the geodesic bundle

$$\eta a_2^\delta x - \left(1 + \eta b_2^\delta\right) y + \eta c_2 [1 - \delta(x^2 + y^2)] = 0, \qquad \eta \in [0, 1], \tag{4.5}$$

with

$$a_2^\delta = \sin(\varphi_1 + \varphi_4) - \delta r_4^2 \sin(\varphi_1 - \varphi_4), \qquad b_2^\delta = \cos(\varphi_1 + \varphi_4) - \delta r_4^2 \cos(\varphi_1 - \varphi_4) - 1,$$
$$c_2 = r_4 \cos \varphi_4$$

in which $r_4 = r_4(r_1)$ is to be inserted.

Multiplication of (4.4) by $\eta c_2$ and subsequent addition to $\xi c_1$ times Eq. (4.5) yields the equation of the line

$$\eta\, x\, \mathcal{B}_\delta(\xi, r_1) = y Q(\xi, \eta, r_1), \tag{4.6}$$

which is a common chord of the two circles (4.4) and (4.5), when we set

$$\begin{aligned} \mathcal{B}_\delta(\xi, r_1) &= r_4 \cos \varphi_4 \left[\xi \left(1 - \delta r_1^2\right) \cos \varphi_2 + (1 - \xi) \cos \varphi_1\right] \\ &\quad + \xi r_1 \cos \varphi_2 \left[\sin(\varphi_1 + \varphi_4) - \delta r_4^2 \sin(\varphi_1 - \varphi_4)\right], \\ Q(\xi, \eta, r_1) &= \xi c_1 - \eta c_2 \sin \varphi_1 + \xi \eta \left(b_1^\delta c_2 + b_2^\delta c_1\right). \end{aligned}$$

Its point of intersection with the circle (4.5) is also the point of intersection of the two circles. Now we can eliminate $x$ from (4.5), using Eq. (4.6). Consequently, we have a quadratic equation in $y$,

$$\frac{\eta^2\mathcal{B}_\delta^2(\xi,r_1)}{y^2}-2\mathcal{C}_\delta(\xi,\eta,r_1)\frac{\eta\mathcal{B}_\delta(\xi,r_1)}{y}-\delta\left[Q^2(\xi,\eta,r_1)+\eta^2\mathcal{B}_\delta^2(\xi,r_1)\right]=0, \quad (4.7)$$

with

$$\begin{aligned}2\mathcal{C}_\delta(\xi,\eta,r_1)=\cos\varphi_1&-\xi\left[\cos\varphi_1-\left(1-\delta r_1^2\right)\cos\varphi_2\right]-\eta\left[\cos\varphi_1-\left(1-\delta r_4^2\right)\cos\varphi_4\right]\\&+\xi\eta\left[\cos\varphi_1-\left(1-\delta r_1^2\right)\cos\varphi_2+\cos(\varphi_1+\varphi_2+\varphi_4)-\left(1-\delta r_4^2\right)\cos\varphi_4\right]\\&-\delta\xi\eta\left[r_1^2\cos(\varphi_1-\varphi_2+\varphi_4)+r_4^2\cos(\varphi_1+\varphi_2-\varphi_4)\right.\\&\left.-\delta r_1^2r_4^2\cos(\varphi_1-\varphi_2-\varphi_4)\right].\end{aligned}$$

Since we need to find the positive root $y$ when $\delta\geq 0$ and the least positive root when $\delta<0$, we obtain from Eqs. (4.7) and (4.6) the following mapping of the unit square on the $(\xi,\eta)$-plane onto the geodesic quadrangle $\mathcal{P}_{r_1}$,

$$x=\frac{Q(\xi,\eta,r_1)}{\mathcal{C}_\delta(\xi,\eta,r_1)+\sqrt{\mathcal{C}_\delta^2(\xi,\eta,r_1)+\delta\left[Q^2(\xi,\eta,r_1)+\eta^2\mathcal{B}_\delta^2(\xi,r_1)\right]}}, \quad (4.8)$$

$$y=\frac{\eta\cdot\mathcal{B}_\delta(\xi,r_1)}{\mathcal{C}_\delta(\xi,\eta,r_1)+\sqrt{\mathcal{C}_\delta^2(\xi,\eta,r_1)+\delta\left[Q^2(\xi,\eta,r_1)+\eta^2\mathcal{B}_\delta^2(\xi,r_1)\right]}}. \quad (4.9)$$

The mapping (4.8)–(4.9) is quasi-isometric, and the inverse mapping is of the form

$$\xi=-\frac{\cos\varphi_1\cdot x+\sin\varphi_1\cdot y}{a_1^\delta x-b_1^\delta y-c_1[1-\delta(x^2+y^2)]},\qquad \eta=\frac{y}{a_2^\delta x-b_2^\delta y+c_2[1-\delta(x^2+y^2)]}. \quad (4.10)$$

Note that in (4.4) and (4.5) in the capacity of $\xi$ and $\eta$ we can take two arbitrary monotonically increasing quasi-isometric functions $\xi(\xi_1)$ and $\eta(\eta_1)$ satisfying the conditions $\xi(0)=\eta(0)=0$ and $\xi(1)=\eta(1)=1$. In particular, we can choose $\xi=\xi(\xi_1)$ and $\eta=\eta(\eta_1)$ in such a way that under the mapping (4.8)–(4.9) the uniform distribution of points on the lower and the left sides of the unit square holds; i.e., the distribution of points on sides $A_1$ and $A_4$ of the geodesic quadrangle is also uniform in a sense of the Euclidean distance. In order to obtain such a mapping it is sufficient to choose $\xi(\xi_1)$ and $\eta(\eta_1)$ as

$$\xi(\xi_1)=\frac{\xi_1\cos\varphi_1}{\xi_1\cos\varphi_1+(1-\xi_1)\cos\varphi_2\left(1+\delta\xi_1r_1^2\right)}, \quad (4.11)$$

$$\eta(\eta_1)=\frac{\eta_1\cos\varphi_1}{\eta_1\cos\varphi_1+(1-\eta_1)\cos\varphi_4\left(1+\delta\eta_1r_4^2\right)}. \quad (4.12)$$

### 4.3. One-Parametric Family of Geodesic Quadrangles

We now proceed by formulating the fundamental relation between angles and sides of $\mathcal{P}$. The angle $\alpha_3=\pi/2+\varphi_3$ at which two geodesics (4.2), (4.3) intersect is defined by (2.6) and $\cos \alpha_3$ is given by

$$\begin{aligned}\left(1+\delta r_1^2\right)\left(1+\delta r_4^2\right)\cos(\pi/2+\varphi_3) = {} & \left[\sin(\varphi_1+\varphi_4)-\delta r_4^2\sin(\varphi_1-\varphi_4)\right]\cos\varphi_2\left(1-\delta r_1^2\right)\\ & +\left[\cos(\varphi_1+\varphi_4)-\delta r_4^2\cos(\varphi_1-\varphi_4)\right]\sin\varphi_2\left(1+\delta r_1^2\right)\\ & -4\delta r_1 r_4\cos\varphi_2\cos\varphi_4. \end{aligned} \tag{4.13}$$

We rewrite (4.13) in the form

$$\begin{aligned} & \sin\frac{\varphi_1+\varphi_2+\varphi_3+\varphi_4}{2}\cos\frac{\varphi_1+\varphi_2-\varphi_3+\varphi_4}{2}\\ & \quad = \delta r_1^2\cos\frac{\varphi_1-\varphi_2+\varphi_3+\varphi_4}{2}\sin\frac{\varphi_1-\varphi_2-\varphi_3+\varphi_4}{2}\\ & \qquad + \delta r_4^2\cos\frac{\varphi_1+\varphi_2+\varphi_3-\varphi_4}{2}\sin\frac{\varphi_1+\varphi_2-\varphi_3-\varphi_4}{2}+2\delta r_1 r_4\cos\varphi_2\cos\varphi_4\\ & \qquad - \delta^2 r_1^2 r_4^2\cos\frac{\varphi_1-\varphi_2-\varphi_3-\varphi_4}{2}\sin\frac{\varphi_1-\varphi_2+\varphi_3-\varphi_4}{2}. \end{aligned} \tag{4.14}$$

From (4.14) it follows that if we construct a convex quadrangle $\mathcal{P}$ for given angles $\alpha_j$, $r_1 \in (r_1^{\min}, r_1^{\max})$ and $\delta = 0$ then its area will be equal to $1/2$.

Equation (4.14) can be considered to be the fundamental relation between sides $r_1, r_4$ and angles $\alpha_1, \ldots, \alpha_4$. For future reference, it is convenient to introduce the notation

$$\begin{aligned} & \gamma = \frac{\varphi_1+\varphi_2+\varphi_3+\varphi_4}{2}, \qquad D = \cos\varphi_2\cos\varphi_4,\\ & C_i = \cos(\gamma-\varphi_i), \qquad S_{ij} = \sin(\gamma-\varphi_i-\varphi_j), \qquad i, j = 1, \ldots, 4, \end{aligned} \tag{4.15}$$

If we determine $\delta$ as follows,

$$\delta = \sin\frac{\varphi_1+\varphi_2+\varphi_3+\varphi_4}{2}, \tag{4.16}$$

then we have from (4.14) a quadratic equation in $r_4$,

$$\left(C_3 - r_1^2 C_2 S_{23}\right)\frac{1}{r_4^2} - 2r_1 D\frac{1}{r_4} + r_1^2 C_1 S_{24}\delta - C_4 S_{34} = 0. \tag{4.17}$$

The solution $r_4 = r_4(r_1)$ for Eq. (4.17) for the case $\delta \geq 0$ is

$$r_4(r_1) = \frac{1}{B_\delta + \sqrt{B_\delta^2 - C_\delta}}, \tag{4.18}$$

where

$$B_\delta = \frac{r_1 D}{C_3 - r_1^2 C_2 S_{23}}, \qquad C_\delta = \frac{r_1^2 C_1 S_{24}\delta - C_4 S_{34}}{C_3 - r_1^2 C_2 S_{23}}$$

provided that following conditions are satisfied:

$$r_1^{\min} < r_1 < r_1^{\max}, \qquad -\frac{\pi}{2} < \varphi_i < \frac{\pi}{2}, \qquad \gamma < \frac{\pi}{2} + \varphi_i \qquad i = 1, \ldots, 4. \tag{4.19}$$

In the case $\delta < 0$ we have to take as $r_4 = r_4(r_1)$ the root of (4.17) that belongs to the interval $(r_4^{\min}, r_4^{\max})$, which we will determine in the next section.

Thus given $r_1$ and four angles $\alpha_1, \ldots, \alpha_4$ we are able to determine the function $r_4 = r_4(r_1)$ and its derivative with respect to $r_1$:

$$\frac{dr_4(r_1)}{dr_1} = -\frac{r_1 C_2 S_{23} + r_4(r_1) D - \delta r_1 r_4^2(r_1) C_1 S_{24}}{r_1 D + r_4(r_1) C_4 S_{34} - \delta r_1^2 r_4(r_1) C_1 S_{24}}. \tag{4.20}$$

In order to use the function $r_4 = r_4(r_1)$ and (4.2)–(4.3) for a quasi-isometric parameterization of elements of family geodesic quadrangles $\mathcal{P}_{r_1}$ we have to find $r_1^{\min}, r_1^{\max}$ and $r_4^{\min}, r_4^{\max}$. But first we shall state some auxiliary statements.

## 4.4. Strict Boundaries for Parameters $r_j$

### 4.4.1. *Geodesic Triangles*

The fundamental relation between angles and sides (4.14) might be used for finding Euclidean lengths for sides not only of geodesic quadrangles, but of geodesic triangles as well.

Let $\mathcal{T}$ be a geodesic triangle on surface of constant curvature $K = 4\delta$ with angles $\beta_i = \pi/2 + \psi_j$ between the sides $B_{j-1}$ and $B_j$, $j = 1, 2, 3$ $(B_0 \equiv B_3)$ and let the vertex of the angle $\beta_1$ be the origin of the parametric plane. We can consider $\mathcal{T}$ as a topological limit of a geodesic quadrangle $\mathcal{P}_{r_4}$ with angles $\alpha_j = \beta_j$, $j = 1, 2, 3$, $\alpha_4 = \pi/2$ and sides $A_1 = B_1$, $A_2 = B_2$, $A_3 + A_4 = B_3$. Under these conditions we can apply (4.14) to find the Euclidean lengths of segments $B_1$ and $B_3$. This argument makes it convenient to introduce a function

$$B(\psi_1, \psi_2, \psi_3, \delta) = \sqrt{\frac{\sin\psi \cos(\psi - \psi_3)}{\delta \cos(\psi - \psi_2)\sin(\psi - \psi_2 - \psi_3)}}, \tag{4.21}$$

where $\psi = (\psi_1 + \psi_2 + \psi_3 + \pi/2)/2$, and $\delta$ is determined by (4.16). By setting $r_4 = 0$, we obtain that the Euclidean length of $B_1$ is equal to $B(\psi_1, \psi_2, \psi_3, \delta)$. In a similar way, the Euclidean length of $B_3$ is $B(\psi_1, \psi_3, \psi_2, \delta)$.

It can happen that one obtains an indeterminate expression of the form 0/0 in (4.21) as $\delta \to 0$ and $\sin\psi \to 0$. To remove this obstacle it is sufficient to define

$$B(\psi_1, \psi_2, \psi_3, 0) = \sqrt{\frac{\cos(\psi - \psi_3)}{\cos(\psi - \psi_2)\sin(\psi - \psi_2 - \psi_3)}}. \tag{4.22}$$

### 4.4.2. *Strict Boundaries for $r_j$, General Case*

Consider one-parametric family of geodesic quadrangles $\mathcal{P}_{r_1}$ with given angles on a surface of constant curvature $K = 4\delta$. With the help of the function (4.21) we are able to find strict boundaries for Euclidean lengths $r_j$, i.e., the end points of the interval to which $r_j$ belongs.

Let us denote by $l(\mathcal{P})$ the number of sides of $\mathcal{P}$, the sum of whose adjacent angles is not less then $\pi$. The value of $l(\mathcal{P})$ can be 0, 1, 2, 3, or 4. Let us consider each case separately.

Let $l(\mathcal{P}_{r_{\sigma(1)}}) = 0$; then for all $\sigma \in \Sigma_4$,

$$r_{\sigma(1)}^{\min} = B\big(\varphi_{\sigma(1)}, \varphi_{\sigma(2)}, -\pi/2, \delta\big), \qquad r_{\sigma(1)}^{\max} = B\big(\varphi_{\sigma(1)}, \pi/2, \varphi_{\sigma(4)}, \delta\big),$$

$$
\begin{aligned}
r_{\sigma(2)}^{\min} &= B\big(\varphi_{\sigma(2)}, \varphi_{\sigma(3)}, -\pi/2, \delta\big), & r_{\sigma(2)}^{\max} &= B\big(\varphi_{\sigma(2)}, \pi/2, \varphi_{\sigma(1)}, \delta\big),\\
r_{\sigma(3)}^{\min} &= B\big(\varphi_{\sigma(3)}, \varphi_{\sigma(4)}, -\pi/2, \delta\big), & r_{\sigma(3)}^{\max} &= B\big(\varphi_{\sigma(3)}, \pi/2, \varphi_{\sigma(2)}, \delta\big),\\
r_{\sigma(4)}^{\min} &= B\big(\varphi_{\sigma(4)}, \varphi_{\sigma(1)}, -\pi/2, \delta\big), & r_{\sigma(4)}^{\max} &= B\big(\varphi_{\sigma(4)}, \pi/2, \varphi_{\sigma(3)}, \delta\big).
\end{aligned}
$$

Let $l(\mathcal{P}_{r_{\sigma(1)}}) = 1$, and $\sigma \in \Sigma_4$ be such that $\varphi_{\sigma(1)} + \varphi_{\sigma(2)} \geq 0$. Then the boundaries will be

$$
\begin{aligned}
r_{\sigma(1)}^{\min} &= 0, & r_{\sigma(1)}^{\max} &= B\big(\varphi_{\sigma(1)}, \pi/2, \varphi_{\sigma(4)}, \delta\big),\\
r_{\sigma(2)}^{\min} &= B\big(\varphi_{\sigma(3)}, \varphi_{\sigma(2)}, -\pi/2, \delta\big), & r_{\sigma(2)}^{\max} &= B\big(\varphi_{\sigma(1)} + \varphi_{\sigma(2)} - \pi/2,\\
& & &\qquad \varphi_{\sigma(3)}, \varphi_{\sigma(4)}, \delta\big),\\
r_{\sigma(3)}^{\min} &= B\big(\varphi_{\sigma(3)}, \varphi_{\sigma(4)}, \varphi_{\sigma(1)} + \varphi_{\sigma(2)} - \pi/2, \delta\big), & r_{\sigma(3)}^{\max} &= B\big(\varphi_{\sigma(3)}, -\pi/2, \varphi_{\sigma(2)}, \delta\big),\\
r_{\sigma(4)}^{\min} &= B\big(\varphi_{\sigma(1)}, \varphi_{\sigma(4)}, -\pi/2, \delta\big), & r_{\sigma(4)}^{\max} &= B\big(\varphi_{\sigma(1)} + \varphi_{\sigma(2)} - \pi/2,\\
& & &\qquad \varphi_{\sigma(4)}, \varphi_{\sigma(3)}, \delta\big).
\end{aligned}
$$

Consider now the case $l(\mathcal{P}_{r_{\sigma(1)}}) \geq 2$. There always exist $\sigma \in \Sigma_4$ such that

$$
\varphi_{\sigma(1)} + \varphi_{\sigma(2)} \geq \varphi_{\sigma(3)} + \varphi_{\sigma(4)}, \qquad \varphi_{\sigma(1)} + \varphi_{\sigma(4)} \geq \varphi_{\sigma(2)} + \varphi_{\sigma(3)}
$$

holds. Then the boundaries for $r_j$ will be

$$
\begin{aligned}
r_{\sigma(1)}^{\min} &= 0, & r_{\sigma(1)}^{\max} &= B\big(\varphi_{\sigma(1)} + \varphi_{\sigma(4)} - \pi/2,\\
& & &\qquad \varphi_{\sigma(2)}, \varphi_{\sigma(3)}, \delta\big),\\
r_{\sigma(2)}^{\min} &= B\big(\varphi_{\sigma(2)}, \varphi_{\sigma(3)}, \varphi_{\sigma(1)} + \varphi_{\sigma(4)} - \pi/2, \delta\big), & r_{\sigma(2)}^{\max} &= B\big(\varphi_{\sigma(1)} + \varphi_{\sigma(2)} - \pi/2,\\
& & &\qquad \varphi_{\sigma(3)}, \varphi_{\sigma(4)}, \delta\big),\\
r_{\sigma(3)}^{\min} &= B\big(\varphi_{\sigma(3)}, \varphi_{\sigma(4)}, \varphi_{\sigma(1)} + \varphi_{\sigma(2)} - \pi/2, \delta\big), & r_{\sigma(3)}^{\max} &= B\big(\varphi_{\sigma(1)} + \varphi_{\sigma(4)} - \pi/2,\\
& & &\qquad \varphi_{\sigma(3)}, \varphi_{\sigma(2)}, \delta\big),\\
r_{\sigma(4)}^{\min} &= 0, & r_{\sigma(4)}^{\max} &= B\big(\varphi_{\sigma(1)} + \varphi_{\sigma(2)} - \pi/2,\\
& & &\qquad \varphi_{\sigma(4)}, \varphi_{\sigma(3)}, \delta\big).
\end{aligned}
$$

### 4.5. Riemannian Metric Induced by the Quasi-Isometric Parameterization of $\mathcal{P}_{r_1}$

Since there exists the analytical representation (4.8)–(4.9) of the quasi-isometric mapping of the unit square $\mathcal{R}$ onto the geodesic quadrangle $\mathcal{P}_{r_1}$, we can find the metric tensor elements $g_{ik}$ in the explicit form. We use abbreviations $\mathcal{B} = \mathcal{B}_\delta(\xi, r_1)$, $Q = Q(\xi, \eta, r_1)$, $\mathcal{C} = \mathcal{C}_\delta(\xi, \eta, r_1)$ and find from (4.8)–(4.9) by differentiation

$$
\begin{aligned}
x_\xi &= \frac{(Q_\xi \mathcal{C} - Q\mathcal{C}_\xi)(\mathcal{C} + Z) + \delta\eta^2 \mathcal{B}(Q_\xi \mathcal{B} - Q\mathcal{B}_\xi)}{Z(\mathcal{C} + Z)^2},\\
y_\xi &= \eta \cdot \frac{(\mathcal{B}_\xi \mathcal{C} - \mathcal{B}\mathcal{C}_\xi)(\mathcal{C} + Z) + \delta Q(\mathcal{B}_\xi Q - \mathcal{B}Q_\xi)}{Z(\mathcal{C} + Z)^2},\\
x_\eta &= \frac{(Q_\eta \mathcal{C} - Q\mathcal{C}_\eta)(\mathcal{C} + Z) + \delta\eta \mathcal{B}^2(\eta Q_\eta - Q)}{Z(\mathcal{C} + Z)^2},\\
y_\eta &= \mathcal{B} \cdot \frac{(\mathcal{C} - \eta\mathcal{C}_\eta)(\mathcal{C} + Z) + \delta Q(Q - \eta Q_\eta)}{Z(\mathcal{C} + Z)^2},
\end{aligned}
\tag{4.23}
$$

where $Z = Z_\delta(\xi, \eta, r_1)$ and $Z_\delta(\xi, \eta, r_1) = \sqrt{\mathcal{C}^2 + \delta[Q^2 + \eta^2\mathcal{B}^2]}$. It is not difficult to verify that

$$\begin{gathered}\mathcal{B}_\xi\mathcal{C} - \mathcal{B}\mathcal{C}_\xi = \frac{1}{2}a_2^\delta\mathcal{A}, \qquad \mathcal{B}_\xi Q - \mathcal{B}Q_\xi = -c_2\mathcal{A}, \qquad Q_\xi\mathcal{C} - Q\mathcal{C}_\xi = \frac{1}{2}\big(1 + \eta b_2^\delta\big)\mathcal{A}, \\ Q_\eta\mathcal{C} - Q\mathcal{C}_\eta = \frac{1}{2}\big(b_1^\delta\xi - \sin\varphi_1\big)\mathcal{B}, \qquad \mathcal{C} - \eta\mathcal{C}_\eta = \frac{1}{2}\big(\cos\varphi_1 + \xi a_1^\delta\big), \qquad Q - \eta Q_\eta = \xi c_1\end{gathered} \tag{4.24}$$

with $\mathcal{A} = \mathcal{A}_\delta(\eta, r_1)$ and

$$\begin{aligned}\mathcal{A}_\delta(\eta, r_1) &= r_1\cos\varphi_2\big[\eta\big(1 - \delta r_4^2\big)\cos\varphi_4 + (1 - \eta)\cos\varphi_1\big] \\ &\quad + \eta r_4\cos\varphi_4\big[\sin(\varphi_1 + \varphi_2) - \delta r_1^2\sin(\varphi_1 - \varphi_2)\big],\end{aligned}$$

and we therefore have

$$\begin{aligned}x_\xi &= \mathcal{A}\left\{\frac{1}{2}\big(1 + \eta b_2^\delta\big)(\mathcal{C} + Z) + \delta\eta^2c_2\mathcal{B}\right\}\cdot Z^{-1}(\mathcal{C} + Z)^{-2}, \\ y_\xi &= \eta\cdot\mathcal{A}\left\{\frac{1}{2}a_2^\delta(\mathcal{C} + Z) - \delta c_2Q\right\}\cdot Z^{-1}(\mathcal{C} + Z)^{-2}, \\ x_\eta &= \mathcal{B}\left\{\frac{1}{2}\big(b_1^\delta\xi - \sin\varphi_1\big)(\mathcal{C} + Z) - \delta\xi\eta c_1\mathcal{B}\right\}\cdot Z^{-1}(\mathcal{C} + Z)^{-2}, \\ y_\eta &= \mathcal{B}\left\{\frac{1}{2}\big(\cos\varphi_1 + \xi a_1^\delta\big)(\mathcal{C} + Z) + \delta\xi c_1Q\right\}\cdot Z^{-1}(\mathcal{C} + Z)^{-2}.\end{aligned} \tag{4.25}$$

We first must characterize functions $\mathcal{A}_\delta(\eta, r_1)$, $\mathcal{B}_\delta(\xi, r_1)$, $\mathcal{C} + Z$, and $Z$. It is an important fact that these functions do not vanish in $\mathcal{R}$. For example, $\mathcal{B}_\delta(\xi_0, r_1) > 0$ and $\mathcal{C} + Z > 0$. This follows from the fact that $y > 0$ for $\eta > 0$ in (4.9). Indeed, if $\mathcal{B}_\delta(\xi_0, r_1) = 0$, $\xi_0 \neq 0$, we would have from (4.9) that $Q(\xi_0, \eta) \equiv 0$ for all $\eta$, while actually the function $Q(\xi_0, \eta)$ has the value $c_1\xi_0 \neq 0$ for $\eta = 0$. The inequality $\mathcal{B}_\delta(\xi_0, r_1) > 0$ obviously implies that $\mathcal{C} + Z > 0$ in the domain $\mathcal{R}$. Denoting by $\mathcal{B}(\xi, r_1, r_4, \varphi_1, \varphi_2, \varphi_4, \delta)$ the function $\mathcal{B}$ with domain of definition (4.1), the function $\mathcal{A}$ is $\mathcal{A}_\delta(\eta, r_1) = \mathcal{B}(\eta, r_4, r_1, \varphi_1, \varphi_4, \varphi_2, \delta)$ and therefore $\mathcal{A}_\delta(\eta, r_1) > 0$ in $\mathcal{R}$. Finally, if $Z_\delta(\xi_0, \eta_0, r_1) = 0$, we would have that tangents to geodesics (4.4) and (4.5) at $(\xi_0, \eta_0)$ are the same, and hence the geodesic (4.4) goes through $(\xi_0, \eta_0)$ and the four different points on the boundary of $\mathcal{P}_{r_1}$. Thus our statement is proved.

Consequently in the capacity of a representative of the class of conformally equivalent Riemannian metrics generated by the mapping (4.8)–(4.9) we can take the metric with the coefficients

$$\begin{aligned}g_{11}^\delta(\xi, \eta, r_1) &= \mathcal{A}^2\Big\{\big[\big(1 + \eta b_2^\delta\big)(\mathcal{C} + Z) + 2\delta\eta^2c_2\mathcal{B}\big]^2\big[a_2^\delta(\mathcal{C} + Z) - 2\delta c_2Q\big]^2\Big\}, \\ g_{22}^\delta(\xi, \eta, r_1) &= \mathcal{B}^2\Big\{\big[\big(b_1^\delta\xi - \sin\varphi_1\big)(\mathcal{C} + Z) - 2\delta\xi\eta c_1\mathcal{B}\big]^2\big[\big(\cos\varphi_1 + \xi a_1^\delta\big) \\ &\qquad \times(\mathcal{C} + Z) + 2\delta\xi c_1Q\big]^2\Big\}, \\ g_{12}^\delta(\xi, \eta, r_1) &= \mathcal{A}\mathcal{B}\big[\big(1 + \eta b_2^\delta\big)(\mathcal{C} + Z) + 2\delta\eta^2c_2\mathcal{B}\big]\big[\big(b_1^\delta\xi - \sin\varphi_1\big)(\mathcal{C} + Z) - 2\delta\xi\eta c_1\mathcal{B}\big] \\ &\qquad + \mathcal{A}\mathcal{B}\big[a_2^\delta(\mathcal{C} + Z) - 2\delta c_2Q\big]\big[\big(\cos\varphi_1 + \xi a_1^\delta\big)(\mathcal{C} + Z) + 2\delta\xi c_1Q\big].\end{aligned} \tag{4.26}$$

Because $\mathcal{A}_\delta(\eta, r_1) \to \mathcal{A}(\eta, r_1)$, $\mathcal{B}_\delta(\xi, r_1) \to \mathcal{B}(\xi, r_1)$, and $2\mathcal{C}_\delta(\xi, \eta, r_1) \to \mathcal{C}(\xi, \eta)$ as $\delta \to 0$, it follows that $g_{ik}^\delta(\xi, \eta, r_1) \to g_{ik}(\xi, \eta, r_1)$ defined by (4.26).

### 4.6. Geodesic Boundary-Fitted Grids

Consider a geodesic quadrangle $\mathcal{P}$ and its parametric representation $x = x(\xi, \eta)$, $y = y(\xi, \eta)$. Let

$$(x_d, y_d) = (x(\xi_d, 0), y(\xi_d, 0)), \qquad (x_u, y_u) = (x(\xi_u, 1), y(\xi_u, 1)), \tag{4.27}$$

$$(x_l, y_l) = (x(0, \eta_l), y(1, \eta_l)), \qquad (x_r, y_r) = (x(0, \eta_r), y(1, \eta_r)), \tag{4.28}$$

be boundary points of $\mathcal{P}$ and $x_d \neq 0$ and $y_l \neq 0$. Now we can find the point of intersection of two geodesic segments that pass through points (4.28) and (4.27), whose equations are

$$a_v x + b_v y - [1 - \delta(x^2 + y^2)] = 0, \tag{4.29}$$

$$a_h x + b_h y + [1 - \delta(x^2 + y^2)] = 0, \tag{4.30}$$

respectively, with

$$\begin{aligned}
a_v &= \left\{y_d - y_u - \delta\left[y_d\left(x_u^2 + y_u^2\right) - y_u\left(x_d^2 + y_d^2\right)\right]\right\}/(x_u y_d - x_d y_u),\\
b_v &= \left\{x_u - x_d + \delta\left[x_d\left(x_u^2 + y_u^2\right) - x_u\left(x_d^2 + y_d^2\right)\right]\right\}/(x_u y_d - x_d y_u),\\
a_h &= \left\{y_l - y_r + \delta\left[y_r\left(x_l^2 + y_l^2\right) - y_l\left(x_r^2 + y_r^2\right)\right]\right\}/(x_l y_r - x_r y_l),\\
b_h &= \left\{x_r - x_l - \delta\left[x_r\left(x_l^2 + y_l^2\right) - x_l\left(x_r^2 + y_r^2\right)\right]\right\}/(x_l y_r - x_r y_l).
\end{aligned}$$

One finds that the point $x = x_{vh}$, $y = y_{vh}$ of intersection of (4.29) and (4.30) is

$$x_{vh} = \frac{-(b_h + b_v)}{a + \sqrt{a^2 + \delta b}}, \qquad y_{vh} = \frac{a_h + a_v}{a + \sqrt{a^2 + \delta b}}, \tag{4.31}$$

where

$$a = \frac{a_h b_v - a_v b_h}{2}, \qquad b = (a_h + a_v)^2 + (b_h + b_v)^2.$$

## 5. VARIATIONAL METHOD FOR THE GENERATION OF QUASI-ISOMETRIC GRIDS

The method by which we solve the BVP for the given Beltrami system is based upon a number of properties of conformal mappings and Theorem 1. The main properties we use are the Riemann theorem, which guarantees the existence and uniqueness of the mapping; boundary properties of conformal mappings; and the Montel variational principle.

### 5.1. A Special Class of Riemannian Manifolds

Consider a geodesic quadrangle $\mathcal{P}$ with given angles $\alpha_i = \varphi_i - \pi/2$ and sides of Euclidean lengths $r_i$, $i = 1, \ldots, 4$ on the surface of constant curvature $K = 4\delta$, $\delta = \sin[(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4)/2]$. Let us embed $\mathcal{P}$ in a set of geodesic quadrangles $\mathcal{P}_r$ with angles $\alpha_1, \ldots, \alpha_4$ depending on a parameter $r = r_1$, $r \in (r_1^{\min}, r_1^{\max})$ such that

$$\mathcal{M}(\mathcal{P}_r) \to 0 \text{ as } r \to r_1^{\min} \qquad \text{and} \qquad \mathcal{M}(\mathcal{P}_r) \to \infty \text{ as } r \to r_1^{\max}. \tag{5.1}$$

We assume that

$$x = x(\xi, \eta, r), \qquad y = y(\xi, \eta, r), \tag{5.2}$$

is a parametric representation $\mathcal{P}_r$ and the metric

$$ds^2 = g_{11}(\xi, \eta, r)\, d\xi^2 + 2g_{12}(\xi, \eta, r)\, d\xi\, d\eta + g_{22}(\xi, \eta, r)\, d\eta^2 \tag{5.3}$$

is a representative of the class of conformally equivalent metrics generated by the mapping (5.2).

Now we can define the Riemannian manifold $\mathcal{N}(g_{ij}(\xi, \eta, r), \mathcal{R})$ with the coordinate domain $\mathcal{R}$, the metric tensor $g_{ij}$, and the parameter $r$.

Then Theorem 1, (5.1), and the Riemann mapping theorem imply the unique existence of the parameter $r^* \in (r_1^{\min}, r_1^{\max})$ and the functions $X^*(\xi, \eta)$, $Y^*(\xi, \eta)$ such that the Riemannian manifold $\mathcal{N}(g_{ij}(\xi, \eta, r^*), \mathcal{R})$ is mapped conformally with respect to the metric (5.3) onto given curvilinear quadrangle $\mathcal{D}$. The mapping $X = X^*(\xi, \eta)$, $Y = Y^*(\xi, \eta)$ is quasi-isometric if all sides of $\mathcal{D}$ are smooth enough (belong to $C^2$) and in addition $\mathcal{P}_{r^*}$ and $\mathcal{D}$ have the same angles [14, 18].

Thus the main problem consists in finding the parameter $r^*$ and a mapping $X^*(\xi, \eta)$, $Y^*(\xi, \eta)$ such that this mapping is conformal with respect to the metric (5.3) with metric tensor $g_{ij}(\xi, \eta, r^*)$.

### 5.2. Functional $\Phi$

Let $\mathcal{N}(g_{ij}(\xi, \eta, r), \mathcal{R})$ be the Riemannian manifold defined above. A class of functions

$$A(\xi, \eta, r) = \frac{g_{22}(\xi, \eta, r)}{g(\xi, \eta, r)}, \qquad B(\xi, \eta, r) = \frac{g_{12}(\xi, \eta, r)}{g(\xi, \eta, r)}, \qquad C(\xi, \eta, r) = \frac{g_{11}(\xi, \eta, r)}{g(\xi, \eta, r)},$$
$$g^2(\xi, \eta, r) = g_{11}(\xi, \eta, r)\, g_{22}(\xi, \eta, r) - g_{12}^2(\xi, \eta, r). \tag{5.4}$$

will be called a class of "admitted" functions. We further introduce the class of admitted mappings $X = X(\xi, \eta)$, $Y = Y(\xi, \eta)$ of the computational region $\mathcal{R}$ onto $\mathcal{D}$ which has the following properties:

1. $X(\xi, \eta)$, $Y(\xi, \eta)$ define a quasi-isometric correspondence between $\partial\mathcal{R}$ and $\partial\mathcal{D}$;
2. $X(\xi, \eta)$, $Y(\xi, \eta)$ can be continued inside $\mathcal{R}$ in such a way that the functional

$$\begin{aligned}\Phi(X, Y, r) = \int_0^1 \int_0^1 & A(\xi, \eta, r)\big(X_\xi^2 + Y_\xi^2\big) - 2B(\xi, \eta, r)(X_\xi X_\eta + Y_\xi Y_\eta) \\ &+ C(\xi, \eta, r)\big(X_\eta^2 + Y_\eta^2\big)\, d\xi\, d\eta\end{aligned} \tag{5.5}$$

is bounded.

The minimum value of the functional is equal to the area $S_{\mathcal{D}}$ of the domain $\mathcal{D}$ [10]. The functions $X^*(\xi, \eta)$, $Y^*(\xi, \eta)$ from the described class and the number $r^*$ that provide the minimum of the functional $\Phi$ give us the desired mapping of the Riemannian manifold $\mathcal{N}(g_{ij}(\xi, \eta, r), \mathcal{R})$ onto $\mathcal{D}$.

### 5.3. Variational Principle

In order to find $X^*(\xi, \eta)$, $Y^*(\xi, \eta)$, and $r^*$ we will construct a minimizing sequence $\{X^n, Y^n, r^n\}$ that has the properties

$$\Phi(X^{n+1}, Y^{n+1}, r^{n+1}) < \Phi(X^n, Y^n, r^n), \qquad \lim_{n\to\infty} \Phi(X^n, Y^n, r^n) = \Phi(X^*, Y^*, r^*) = S_{\mathcal{D}}.$$

At the beginning of the minimization process we assume that the functions $X^n(\xi, \eta)$, $Y^n(\xi, \eta)$, and $r^n$ are known to us. The first step requires us to obtain $r^{n+1}$ such that

$$\Phi(X^n, Y^n, r^{n+1}) < \Phi(X^n, Y^n, r^n).$$

The construction of the sequence $\{r^n\}$ is based on the fact that the parametric representation $\mathcal{P}_r$, that is, the mapping (5.2), is defined on some neighborhood of the set $\mathcal{R}$ as well. This allows us to embed the mapping (5.2), having $r$ fixed, into the family of mappings depending on a parameter $\varepsilon$ in the following way,

$$x^\varepsilon(\xi, \eta, r) = x((1+\varepsilon\mu)\xi, (1-\varepsilon\nu)\eta, r), \tag{5.6}$$

$$y^\varepsilon(\xi, \eta, r) = y((1+\varepsilon\mu)\xi, (1-\varepsilon\nu)\eta, r), \tag{5.7}$$

where $\varepsilon$ and constants $\mu$, $\nu$ satisfy the following inequalities: $\mu\nu > 0$,

$$r_1^{\min} < x(1+\varepsilon\mu, 0, r) < r_1^{\max}, \qquad r_4^{\min} < \sqrt{x^2(0, 1-\varepsilon\nu, r) + y^2(0, 1-\varepsilon\nu, r)} < r_4^{\max}.$$

Every mapping (5.6)–(5.7) is quasi-isometric, and the boundary $\partial\mathcal{R}$ goes over into a geodesic quadrangle $\mathcal{P}_r^\varepsilon$. From the Mantel variational principle [17], in a manner similar to that given in [6] we obtain that the conformal modulus of the geodesic quadrangle $\mathcal{P}_r^\varepsilon$ satisfies the inequality $\mathcal{M}(\mathcal{P}_r^\varepsilon) > \mathcal{M}(\mathcal{P}_r^{\bar\varepsilon})$ for every $\varepsilon > \bar\varepsilon$. Consequently for every fixed $r$ there exists a unique $\varepsilon$ such that $\mathcal{M}(\mathcal{P}_r^\varepsilon) = \mathcal{M}(\mathcal{D})$. Moreover, the family of mappings (5.6)–(5.7) generates the family of metrics with the metric tensor elements

$$\begin{aligned}
g_{11}^\varepsilon(\xi, \eta, r) &= g_{11}(\xi, \eta, r)(1+\varepsilon\mu)^2,\\
g_{22}^\varepsilon(\xi, \eta, r) &= g_{22}(\xi, \eta, r)(1-\varepsilon\nu)^2,\\
g_{12}^\varepsilon(\xi, \eta, r) &= g_{12}(\xi, \eta, r)(1+\varepsilon\mu)(1-\varepsilon\nu).
\end{aligned}\tag{5.8}$$

Using (5.8) and (5.4) we can find $A^\varepsilon(\xi, \eta, r)$, $B^\varepsilon(\xi, \eta, r)$, $C^\varepsilon(\xi, \eta, r)$, such that $A^0(\xi, \eta, r) = A(\xi, \eta, r)$, $B^0(\xi, \eta, r) = B(\xi, \eta, r)$, $C^0(\xi, \eta, r) = C(\xi, \eta, r)$, and substituting these into (5.5), we will obtain the functional $\Phi^\varepsilon(X, Y, r)$. It is easy to verify that having $X$, $Y$, and $r$ fixed at the stationary point of the functional $\Phi^\varepsilon(X, Y, r)$ the equality

$$\varepsilon = \frac{\Lambda - V}{\nu\Lambda + \mu V} \tag{5.9}$$

must hold, where

$$\Lambda^2 = \int_0^1\int_0^1 A(\xi, \eta, r)\left(X_\xi^2 + Y_\xi^2\right) d\xi\, d\eta, \qquad V^2 = \int_0^1\int_0^1 C(\xi, \eta, r)\left(X_\eta^2 + Y_\eta^2\right) d\xi\, d\eta,$$

and in the capacity of constants $\mu, \nu$ we can use $\mu = 1/\sqrt{g_{11}(1, 0, r)}$ and $\nu = -(d/dr_1)r_4(r_1)/\sqrt{g_{22}(0, 1, r)}$. Denoting by $\varepsilon = \varepsilon(X, Y, r)$ the function (5.9) and taking $\varepsilon = \varepsilon(X^n, Y^n, r^n)$, we can project $A^{\varepsilon}(\xi, \eta, r^n)$, $B^{\varepsilon}(\xi, \eta, r^n)$, $C^{\varepsilon}(\xi, \eta, r^n)$ onto the class of admitted functions $A$, $B$, $C$. In order to do this it is sufficient to set $r^{n+1} = r^n + \varepsilon$ and obtain $A = A(\xi, \eta, r^{n+1})$, $B = B(\xi, \eta, r^{n+1})$, and $C = C(\xi, \eta, r^{n+1})$ using the formulas (5.4).

On the second step for computation of new approximation $X^{n+1}$, $Y^{n+1}$ such that

$$\Phi(X^{n+1}, Y^{n+1}, r^{n+1}) < \Phi(X^n, Y^n, r^{n+1}),$$

we use received $A$, $B$, $C$ as coefficients of the elliptic equations that represent the variational Euler–Lagrange equations for the functional (5.5) being minimized on $X$ and $Y$:

$$-\frac{\partial}{\partial \xi} A \frac{\partial X}{\partial \xi} - \frac{\partial}{\partial \eta} C \frac{\partial X}{\partial \eta} + \left( \frac{\partial}{\partial \xi} B \frac{\partial X}{\partial \eta} + \frac{\partial}{\partial \eta} B \frac{\partial X}{\partial \xi} \right) = 0, \tag{5.10}$$

$$-\frac{\partial}{\partial \xi} A \frac{\partial Y}{\partial \xi} - \frac{\partial}{\partial \eta} C \frac{\partial Y}{\partial \eta} + \left( \frac{\partial}{\partial \xi} B \frac{\partial Y}{\partial \eta} + \frac{\partial}{\partial \eta} B \frac{\partial Y}{\partial \xi} \right) = 0. \tag{5.11}$$

The solution $X$, $Y$ of the system (5.10)–(5.11) with appropriate boundary conditions can be used as a new approximation $X^{n+1}$, $Y^{n+1}$.

Steps 1 and 2 are to be repeated till the desired accuracy of determining the solution of the variational problem is achieved.

### 5.4. A Finite-Difference Approximation of the Functional $\Phi$

Let us assume that $X_{ij} = X(i/I, j/J)$, $Y_{ij} = Y(i/I, j/J)$, $i = 0, \ldots, I$, $j = 0, \ldots, J$. We call image of the rectangle with vertices $(i/I, j/J)$, $((i-1)/I, j/J)$, $((i-1)/I, (j-1)/J)$, $(i/I, (j-1)/J)$ under the mapping (1.1) "a cell with number $(i, j)$," $1 \le i \le I$, $1 \le j \le J$. In every cell we assume functions $A$, $B$, $C$ and $E = (X_\xi)^2 + (Y_\xi)^2$, $F = X_\xi X_\eta + Y_\xi Y_\eta$, $G = (X_\eta)^2 + (Y_\eta)^2$ to be constant and defined as a set

$$\{A_{ij}, B_{ij}, C_{ij}, E_{ij}, F_{ij}, G_{ij} \; i = 1, \ldots, I, \; j = 1, \ldots, J\}.$$

Thus, the finite-difference approximation of a functional $\Phi$ has the form

$$\hat{\Phi} = \sum_{i=1}^{I} \sum_{j=1}^{J} [A_{ij} E_{ij} - 2B_{ij} F_{ij} + C_{ij} G_{ij}].$$

We will calculate derivatives $X_\xi$, $X_\eta$, $Y_\xi$, $Y_\eta$ in every cell with the help of equations

$$\{X_\xi\}_{ij} = \frac{1}{2}(X_{ij} - X_{i-1,j} + X_{i,j-1} + X_{i-1,j-1}),$$
$$\{X_\eta\}_{ij} = \frac{1}{2}(X_{ij} - X_{i,j-1} + X_{i-1,j} + X_{i-1,j-1}),$$

in which $\Delta\xi = \Delta\eta = 1$, since $\hat{\Phi}$ does not depend on $\Delta\xi$ and $\Delta\eta$, and $\{Y_\xi\}_{ij}$, $\{Y_\eta\}_{ij}$ can be obtained simply by the substitution of $Y$ instead of $X$.

### 5.5. The Algorithm

Below we give a simple explicit algorithm for the generation of the quasi-isometric or quasi-conformal grids inside a region $\mathcal{D}$ if the boundary points are fixed. We do not focus on numerical methods for elliptic equations. Many of them can be found in several books, for example [9].

1. Define the distribution of boundary points of $\mathcal{D}$:

$$\{(X_{i0}, Y_{i0}), (X_{iJ}, Y_{iJ}), i = 0, \dots, I, (X_{0j}, Y_{0j}), (X_{Ij}, Y_{Ij}), j = 0, \dots, J\}.$$

2. Define initial interior grid points of $\mathcal{D}$: $\{(X_{ij}^{\text{old}}, Y_{ij}^{\text{old}}), i = 1, \dots, I-1, j = 1, \dots, J-1\}$.
3. Define angles $\alpha_1, \dots, \alpha_4$ satisfying the condition (1.5) and calculate $r_k^{\min}, r_k^{\max}$ for $k = 1, \dots, 4$—boundaries of Euclidean lengths of sides of quadrangles from the one-parameter family $\mathcal{P}_{r_1}$.
4. Define the first approximation for $r_1 = r_1^{\text{old}}$ from the interval $(r_1^{\min}, r_1^{\max})$.
5. Define the initial distribution of boundary points on $\mathcal{P}_{r_1^{\text{old}}}$:

$$\left\{\left(x_{i0}^{\text{old}}, y_{i0}^{\text{old}}\right), \left(x_{iJ}^{\text{old}}, y_{iJ}^{\text{old}}\right), i = 0, \dots, I, \left(x_{0j}^{\text{old}}, y_{0j}^{\text{old}}\right), \left(x_{Ij}^{\text{old}}, y_{Ij}^{\text{old}}\right), j = 0, \dots, J\right\}.$$

6. Construct a geodesic grid $\{(x_{ij}, y_{ij}), i = 0, \dots, I, j = 0, \dots, J\}$ in $\mathcal{P}_{r_1^{\text{old}}}$ using the formula (4.31).
7. Moving each vertex of the cell $(i, j)$ of the geodesic quadrangle $\mathcal{P}_{r_1^{\text{old}}}$ to the origin by motions (2.7) to calculate parameters $r_1^{ij}, r_4^{ij}, \varphi_1^{ij}, \varphi_2^{ij}, \varphi_4^{ij}$ of a geodesic cell and calculate coefficients $A_{ij}$, $B_{ij}$, and $C_{ij}$ of the functional using the formula (4.25).
8. Using $A_{ij}$, $B_{ij}$, and $C_{ij}$ find new coordinates of interior grid points $(X_{ij}^{\text{new}}, Y_{ij}^{\text{new}})$ in the physical domain $\mathcal{D}$ by means of applying the iterative method

$$z_{ij}^{\text{new}} = z_{ij}^{\text{old}} + d_{ij}\frac{\partial \Phi}{\partial z_{ij}}, \qquad i = 1, \dots, I-1, j = 1, \dots, J-1,$$

where $z_{ij}^{\text{new}}$ is either $X_{ij}^{\text{new}}$ or $Y_{ij}^{\text{new}}$ and

$$d_{ij} = \frac{\theta}{2(A_{ij} + C_{ij}) + |B_{ij}|}.$$

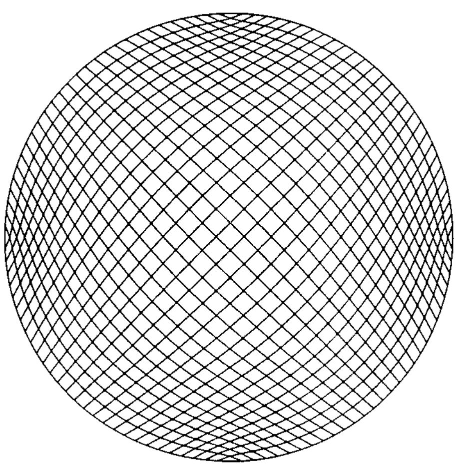

**FIG. 1.** Quasi-conformal grid inside a circle with fixed boundary points and no adaptation.

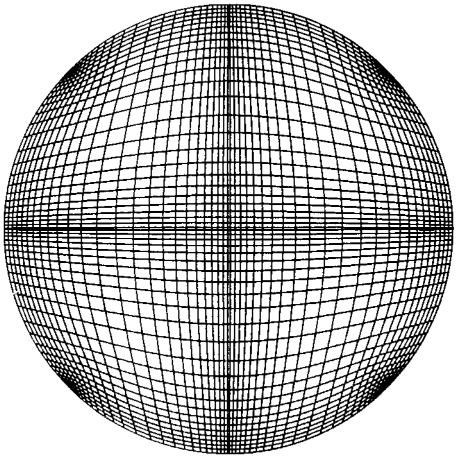

**FIG. 2.** Quasi-conformal grid inside a circle with fixed boundary points and adaptation.

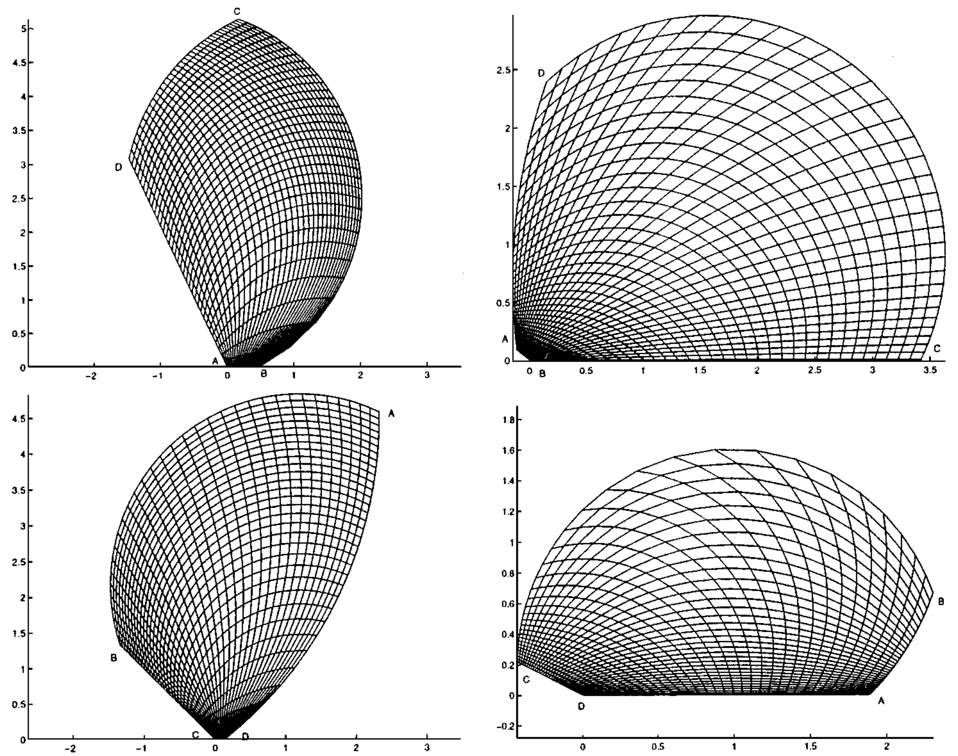


**FIG. 3.** A geodesic quadrangle on a surface of constant positive curvature. All vertices are placed at the origin in turns.

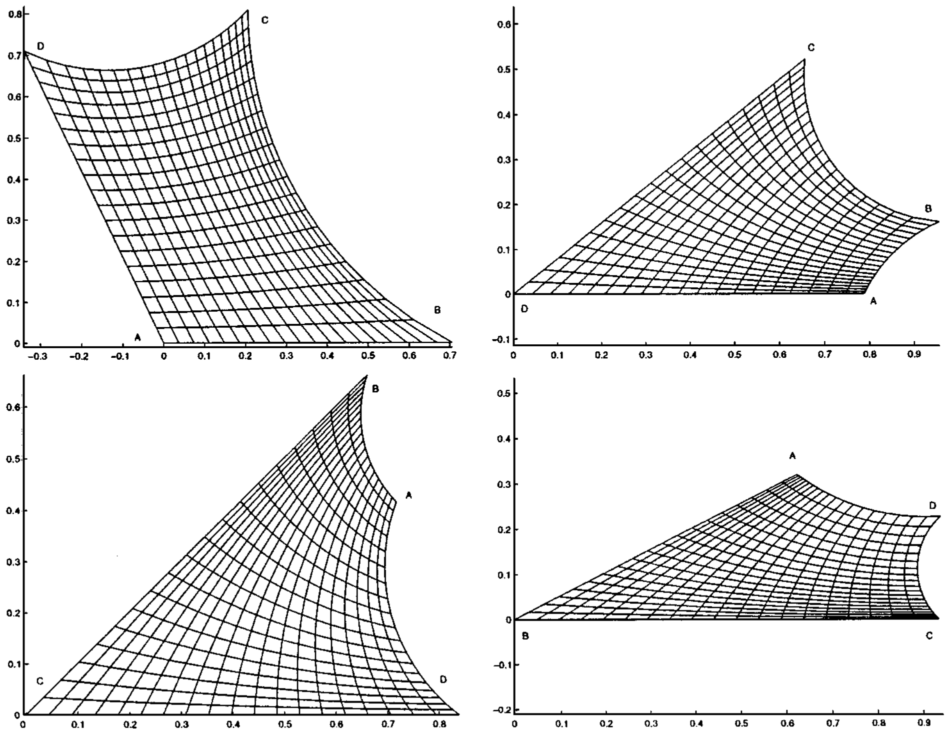


**FIG. 4.** A geodesic quadrangle on a surface of constant negative curvature. All vertices are placed at the origin in turns.

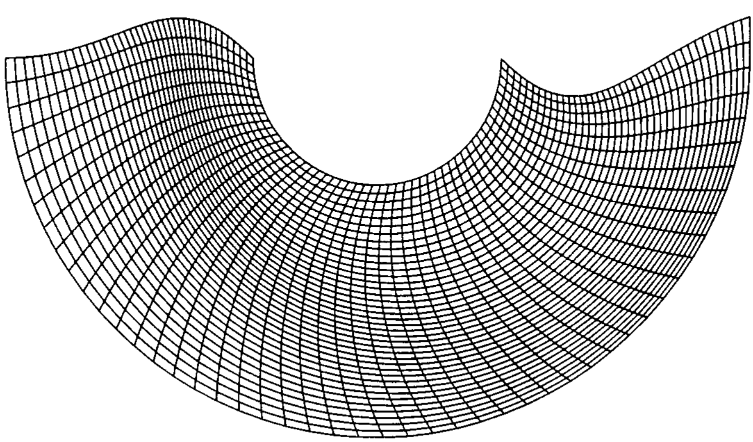

**FIG. 5.** Quasi-isometric grid in a test domain with negative angle defect.

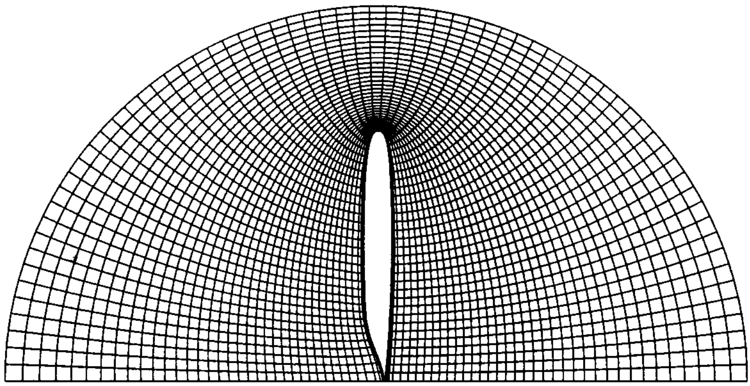

**FIG. 6.** Quasi-isometric grid around an airfoil with no adaptation and free boundary points.

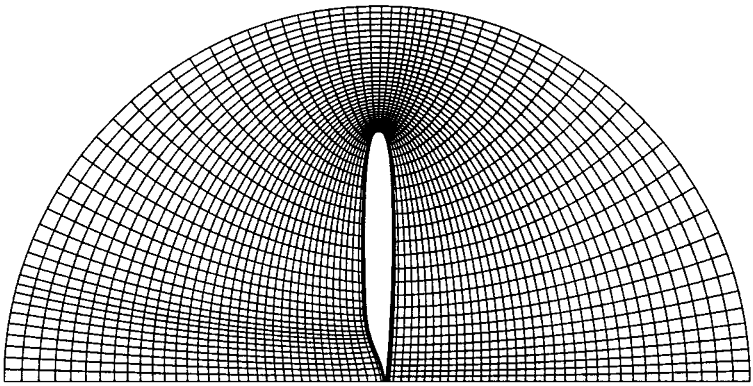

**FIG. 7.** Quasi-isometric grid around the same airfoil as in Fig. 6 with adaptation and fixed boundary points.

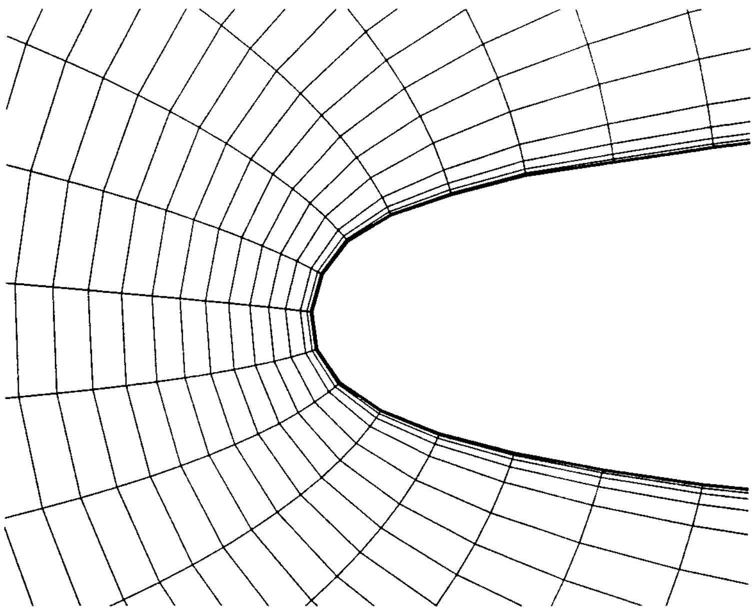

**FIG. 8.** Part of the grid from Fig. 7.

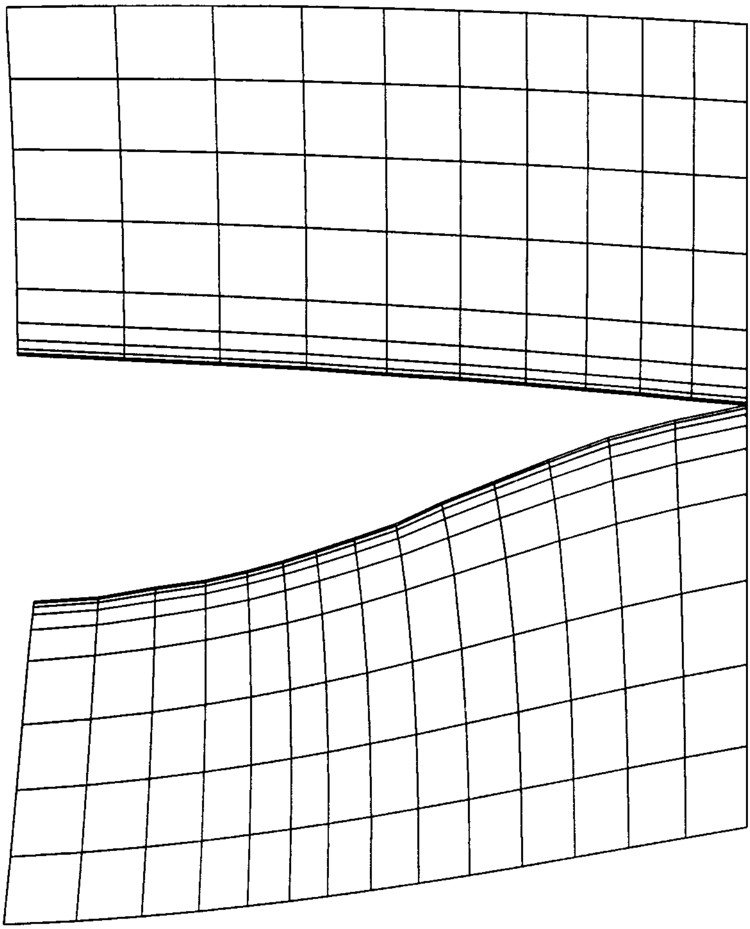

**FIG. 9.** Another part of the grid from Fig. 7, showing parallelograms in corners of the domain.

Here $\theta$ is a damper, $0 < \theta < 2$. The derivatives $\partial\Phi/\partial z_{ij}$ can be obtained from formulas

$$\frac{\partial \Phi}{\partial z_{ij}} = \sum_{k=1}^{4} [U_k(i, j) z_{ij} - V_k(i, j)],$$

where

$$\begin{aligned}
U_1(i, j) &= A_{ij} - B_{ij} + C_{ij},\\
U_2(i, j) &= A_{i+1,j} + B_{i+1,j} + C_{i+1,j},\\
U_3(i, j) &= A_{i+1,j+1} - B_{i+1,j+1} + C_{i+1,j+1},\\
U_4(i, j) &= A_{i,j+1} + B_{i,j+1} + C_{i,j+1},\\
V_1(i, j) &= A_{ij} z_{i-1,j} - B_{ij} z_{i-1,j-1} + C_{ij} z_{i,j-1},\\
V_2(i, j) &= A_{i+1,j} z_{i+1,j} + B_{i+1,j} z_{i+1,j-1} + C_{i+1,j} z_{i,j-1},\\
V_3(i, j) &= A_{i+1,j+1} z_{i+1,j} - B_{i+1,j} z_{i+1,j+1} + C_{i+1,j+1} z_{i,j+1},\\
V_4(i, j) &= A_{i,j+1} z_{i-1,j} + B_{i,j+1} z_{i-1,j+1} + C_{i,j+1} z_{i,j+1}.
\end{aligned}$$

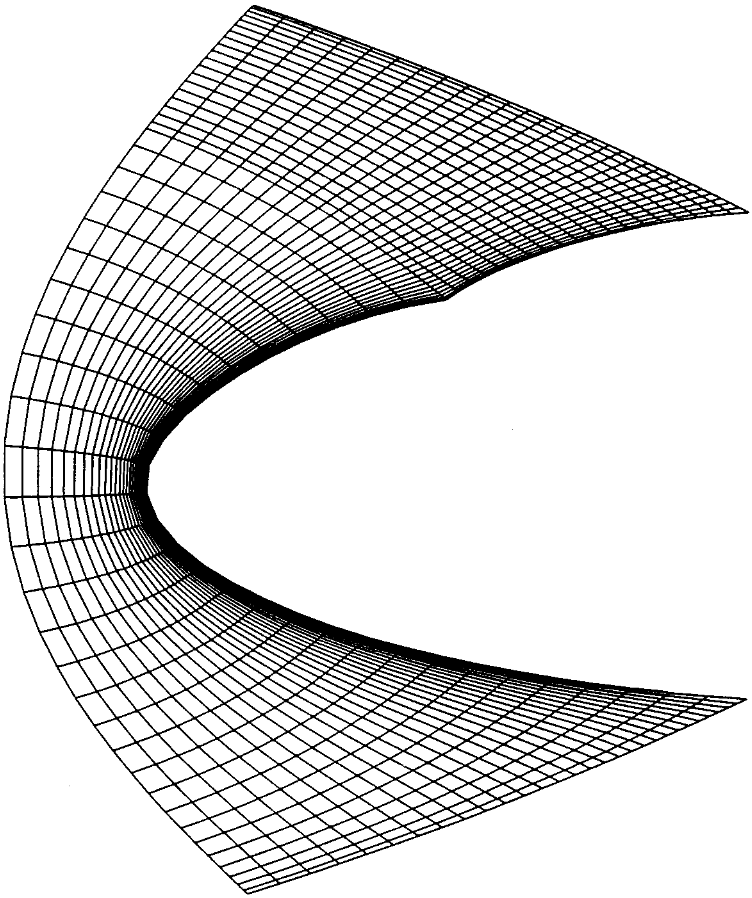

**FIG. 10.** A quasi-conformal grid around an airplane nose. Points on the airplane surface are fixed, and points in the opposite boundary are forced to keep "same as opposite" distribution.

9. Obtain coordinates of perturbed boundary points

$$\bar{X}_{ij} = X_{ij} + s_{ij} p_{ij}, \qquad \bar{Y}_{ij} = Y_{ij} + s_{ij} q_{ij},$$

where $(p_{ij}, q_{ij})$ is a unit tangent vector at the boundary point $(X_{ij}, Y_{ij})$,

$$s_{ij} = (\Delta X_{ij} p_{ij} + \Delta Y_{ij} q_{ij})\, d_{ij}, \qquad d_{ij} > 0,$$

and in the following $z$ is either $X$ or $Y$:

$$\Delta z_{0j} = \frac{V_2(0, j) + V_3(0, j)}{U_2(0, j) + U_3(0, j)}, \qquad \Delta z_{Ij} = \frac{V_1(I, j) + V_4(I, j)}{U_1(I, j) + U_4(I, j)},$$

$$\Delta z_{i0} = \frac{V_4(i, 0) + V_3(i, 0)}{U_4(i, 0) + U_3(i, 0)}, \qquad \Delta z_{iJ} = \frac{V_1(i, J) + V_2(i, J)}{U_1(i, J) + U_2(i, J)}.$$

10. Having $X_{ij}^{\text{new}}$, $Y_{ij}^{\text{new}}$, calculate $E_{ij}$, $G_{ij}$ and obtain

$$\varepsilon = \frac{\hat{\Lambda} - \hat{V}}{\nu\hat{\Lambda} + \mu\hat{V}},$$

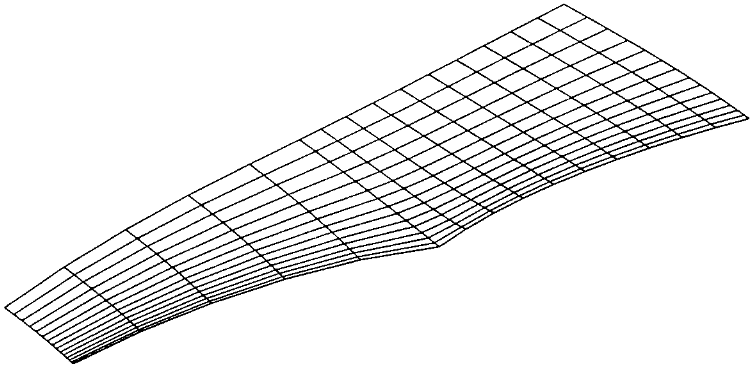

**FIG. 11.** Part of the grid from the Fig. 10, with a singularity on one side.

where

$$\hat{\Lambda}^2 = \sum_{i=1}^{I}\sum_{j=1}^{J} A_{ij}E_{ij}, \qquad \hat{V}^2 = \sum_{i=1}^{I}\sum_{j=1}^{J} C_{ij}G_{ij}$$

and

$$\mu = 1\Big/\sqrt{g_{11}(1, 0, r^{\text{old}})}, \qquad \nu = -\frac{d}{dr_1}r_4(r_1)\Big/\sqrt{g_{22}(0, 1, r^{\text{old}})}.$$

11. Find a new parameter $r_1^{\text{new}} = r_1^{\text{old}} + \varepsilon$ and verify that $r_1^{\min} < r_1^{\text{new}} < r_1^{\max}$.

12. Using the correspondence between $\mathcal{P}_{r_1^{\text{old}}}$ and $\mathcal{D}$ obtain prototypes of perturbed boundary points $(\overline{X}_{ij}, \overline{Y}_{ij})$ on $\mathcal{P}_{r_1^{\text{new}}}$,

$$\left\{\left(x_{i0}^{\text{new}}, y_{i0}^{\text{new}}\right), \left(x_{iJ}^{\text{new}}, y_{iJ}^{\text{new}}\right), i = 0, \ldots, I, \left(x_{0j}^{\text{new}}, y_{0j}^{\text{new}}\right), \left(x_{Ij}^{\text{new}}, y_{Ij}^{\text{new}}\right), j = 0, \ldots, J\right\}.$$

13. Steps 6–11 are to be repeated until the desired accuracy of solution of the variational problem is achieved.

### 5.6. Examples

In Figs. 1–11 we provide some examples of the work of the algorithm.

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